6.5: Impulse functions

Unit impulse function = Dirac delta function is a generalized function with the properties

$$\delta(t) = 0, \quad t \neq 0 \qquad \int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\mathcal{L}(\delta(t - t_0)) = e^{-st_0}$$

Let $d_k(t) = \begin{cases} \frac{1}{2k} & -k < t < k \\ 0 & t \leq -k \text{ or } t \geq k \end{cases}$

Note $lim_{k\to 0} d_k(t) = 0$ if $t \neq 0$

and $lim_{k\to 0} \int_{-\infty}^{\infty} d_k(t) = lim_{k\to 0} 1 = 1 = \int_{-\infty}^{\infty} \delta(t)dt$

$$\mathcal{L}(\delta(t - t_0)) = lim_{k\to 0}\mathcal{L}(d_k(t - t_0))$$

$$= lim_{k\to 0} \int_0^{\infty} e^{-st} d_k(t - t_0)dt$$

$$= lim_{k\to 0} \frac{1}{2k} \int_{t_0-k}^{t_0+k} e^{-st}dt$$

$$= lim_{k\to 0} \frac{-1}{2sk} e^{-st}\Big|_{t_0-k}^{t_0+k}$$

$$= lim_{k\to 0} \frac{1}{2sk} e^{-st_0}(e^{sk} - e^{-sk})$$

$$= lim_{k\to 0} \frac{sinh(sk)}{sk} e^{-st_0}$$

$$= lim_{k\to 0} \frac{scosh(sk)}{s} e^{-st_0} \;=\; e^{-st_0}$$

$sin(t) = \frac{e^{it} - e^{-it}}{2i}$ $\qquad$ $cos(t) = \frac{e^{it} + e^{-it}}{2}$

$sinh(t) = \frac{e^{t} - e^{-t}}{2}$ $\qquad$ $cosh(t) = \frac{e^{t} + e^{-t}}{2}$

$[sinh(t)]' =$ $\qquad$ $[cosh(t)]' =$

$sinh(0) = \frac{e^0 - e^0}{2} = 0$ $\qquad$ $cosh(0) = \frac{e^0 + e^0}{2} = 1$

Intro to Group Theory

Define the $\cdot$ product on R^2 by

$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 - y_1x_2)$

Note $\cdot$ is

1.) commutative:
$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 - y_1x_2)$
$= (x_2x_1 - y_2y_1, x_2y_1 - y_2x_1) = (x_2, y_2) \cdot (x_1, y_1)$

2.) associative: $(f \cdot g) \cdot h = f \cdot (g \cdot h)$

3.) distributive w.r.t +: $f \cdot (g_1 + g_2) = f \cdot g_1 + f \cdot g_2$

4.) $(x_1, y_1) \cdot (0, 0) = (0, 0)$

Note $(0, 1) \cdot (0, 1) = (-1, 0)$

6.6: The Convolution Integral

Defn: The convolution of f and g is the function $f*g$ defined by

$$(f*g)(t) = \int_0^t f(t-s)g(s)ds = \int_0^t f(x)g(t-x)dx$$

Note $*$ is

1.) commutative: $f*g = g*f$

2.) associative: $(f*g)*h = f*(g*h)$

3.) distributive w.r.t +: $f*(g_1+g_2) = f*g_1 + f*g_2$

4.) $f*0 = 0*f = 0$

Example: $cos(t)*1 =$

Example: $t*t \ngeq 0$

Thm: $\mathcal{L}((f*g)(t)) = \mathcal{L}(f(t)) \cdot \mathcal{L}(g(t))$

Proof:

$\mathcal{L}(f(t))\mathcal{L}(g(t)) = \int_0^\infty e^{-sy}f(y)dy \int_0^\infty e^{-sx}g(x)dx$

$= \int_0^\infty [\int_0^\infty e^{-sy}f(y)dy]e^{-sx}g(x)dx$

$= \int_0^\infty [\int_0^\infty e^{-sy}f(y)e^{-sx}g(x)dy]dx$

$= \int_0^\infty [\int_0^\infty e^{-s(y+x)}f(y)g(x)dy]dx$

$= \int_0^\infty [\int_0^\infty e^{-s(y+x)}f(y)g(x)dx]dy$

Let $t = x+y$, $dt = dx$

$= \int_0^\infty [\int_y^\infty e^{-st}f(y)g(t-y)dt]dy$

$= \int_0^\infty [\int_0^t e^{-st}f(y)g(t-y)dy]dt$

$= \int_0^\infty e^{-st}[\int_0^t f(y)g(t-y)dy]dt$

$= \int_0^\infty e^{-st}(f*g)(t)dt$

$= \mathcal{L}(f*g)$

Example: $\mathcal{L}^{-1}(\frac{1}{s(s-a)}) =$