Analysis and Probability

Wavelets, Signals, Fractals

Palle E. T. Jorgensen

with graphics by Brian Treadway

41 figures and illustrations

Department of Mathematics, The University of Iowa, 14 MacLean Hall, Iowa City, IA 52242-1419, U.S.A.
E-mail address: `jorgen@math.uiowa.edu`
URL: `http://www.math.uiowa.edu/~jorgen/`

Brief description: This book, combining analysis and tools from mathematical probability focuses on a systematic and novel presentation of a recent trend in pure and applied mathematics. The emphasis is on the unity of basis constructions and their expansions; and on their use in several areas, from wavelets to fractals; bases which are computationally efficient. The book brings together tools from engineering and from math, especially from signal and image processing, and from harmonic analysis and operator theory in math. The presentation, including exercises is aimed at graduate students, and at users from a diverse spectrum of applications. Among other things, the presentation stresses:

- a hands on approach for students, including tutorials and many exercises,
- a generous amount motivation for each part of the book,
- new pedagogical features which makes the book useful in teaching,
- includes more than 41 figures with captions, illustrating the main ideas, plus engineering diagrams, graphic rendition of algorithms, and separate illustrations,
- separate sections in the book explain engineering terms to mathematicians, and operator theory to engineers.
- Each chapter concludes with a helpful guide to the literature allowing students to follow up on the topics in the book.

Preface

> *From its shady beginnings devising gambling strategies and counting corpses in medieval London, probability theory and statistical inference now emerge as better foundations for scientific models, especially those of the process of thinking and as essential ingredients of theoretical mathematics, even the foundations of mathematics itself.* —David Mumford

An apology. You ask: "Why all the fuss?" — Wavelets, signals, fractals? Isn't all of this merely a fad? Or a transient popularity trend? And what's the *probability* part in the book all about? And non-commuting operators? As for bases in linear spaces, what's wrong with Gram–Schmidt?

You may think: "Fourier has served us well for ages; so why do we need all the other basis functions?" — Wavelets and so on? — And why engineering topics in a math course? And the pictures? Are they really necessary?

And engineering topics: Signal processing and image processing. — Yes, technology is lovely, but why not leave this to the engineers?

Response: The links between mathematics and engineering are much deeper than the fact that we mathematicians teach service courses for engineers. Our bread and butter!

Mathematics draws ideas and strengths from the outside world, and the connections to parts of engineering have been a boon to math: From signal processing to wavelet analysis! That is true even if we forget about all of the practical applications emerging from these connections. Without inspiration from the neighboring sciences math would in all likelihood become rather sterile, and overly formal. I see opportunities at crossroads. In this book you will see the benefits math is reaping from trends and topics in engineering. It is witnessed in a striking way by exciting developments in wavelets. From wavelets we see how notions of scale-similarity can be exploited in basis computations that use tricks devised for signal processing. Just open the book and glance at some of the wavelet functions. At the same time, the key notion of self-similarity, such as the scale-similarity used everywhere for wavelets, is essential to our understanding of fractals: Fern-like pictures that look the same at small and at large scales. One problem in the generation of wavelet bases is selecting the "nice" (here this means differentiable) wavelets among huge families of fractal-looking (non-smooth, or singular) functions. L^2-functions can be very "bad" indeed!! Computers generate the good and the bad, and we are left with the task of sorting these out and making selections. We will see (directly from large libraries of pictures) that mathematical wavelet machines are more likely to spit out bad functions unless they are told from the intrinsic math where to concentrate the search.

These things, wavelets, signals, and fractals, have caught our attention in recent decades, but the mathematical part of this has roots back at least a hundred years, for example, to Alfred Haar and to Oliver Heaviside at the turn of the last century. From Haar we have the first wavelet basis, and with Heaviside we see the beginning of signal analysis. It is unlikely that either one knew about the other. Ironically, at the time (1909), Haar's paper had little impact and was hardly noticed, even on the small scale of "notice" that is usually applied to math papers. Haar's wonderful wavelet only began to draw attention in the mid-nineteen-eighties when the connections to modern signal processing became much better understood. These connections certainly served as a main catalyst in what are now known as wavelet tools in pure and applied mathematics. But at the outset, the pioneers in wavelets had to "rediscover" a lot of stuff from signal processing: frequency bands, high-pass, low-pass, analysis and synthesis using down-sampling, and up-sampling, reconstruction of signals, resolution of images; all tools that have wonderful graphics representations in the engineering literature.

But still, why aren't Fourier's basis, and his lovely integral decomposition, good enough? Many reasons: Fourier's method has computational drawbacks. This wasn't evident until computers became common and began to play important roles in applied and theoretical work. But expansion of functions or signals into basis decompositions (called "analysis" in signal processing) involves basis coefficients (Fourier coefficients and so on), and if we are limited to Fourier's bases, then the computation of the coefficients must by necessity rely on integration. Computers can't integrate! Hmmm! Well, not directly. The problem must first be discretized. And there is need for a more direct and algorithmic approach. Hence the wavelet algorithm! In any case, algorithms are central in math even if you don't care about computers. And it is the engineering connections that inspired the most successful algorithms in our subject.

Multiresolutions. If you stop to think about it, it is really ironic that Haar's wavelet basis was missed for so long. It is especially ironic since it had in it implicitly the key idea which got wavelet mathematics started on a roll twenty years ago with Yves Meyer, Ingrid Daubechies, Stephane Mallat, and others; namely the idea of a multiresolution. In that respect Haar was ahead of his time. In 1909, Haar's measure on non-abelian groups became much more widely known.

Yet, returning to wavelets, the word "multiresolution" suggests a connection to optics from physics. So that should have been a hint to mathematicians to take a closer look at trends in signal and image processing! Moreover, even staying within math, it turns out that as a general notion this very same idea of a "multiresolution" has long roots in math, even in such modern and pure areas as operator theory and Hilbert-space geometry. And in probability theory and in dynamics, A.N. Kolmogorov and J. Doob had already long ago identified martingales, again close cousins of multiresolutions. Looking even closer at these interconnections, we can now recognize scales of subspaces (so-called multiresolutions) in classical algorithmic construction of orthogonal bases in inner-product spaces, now taught in lots of math courses under the name of the Gram–Schmidt algorithm. Indeed, a closer look at good old Gram–Schmidt reveals that it is a matrix algorithm. Hence new mathematical tools involving non-commutativity! Obviously, function spaces are infinite-dimensional. Since Gram–Schmidt is recursive, it doesn't stop; at least not until we tell it to stop. We do that when the basis expansion has achieved a "good enough" approximation to the true function, or the true signal or image which is being analyzed.

Approximation? So we must retain the "significant" terms in an analysis expansion and throw out the other terms! To know which is "significant", thresholds must be assigned, and probabilities, and even entropy, from information theory must be used.

A Wiener process, carried to the limit, gives a nowhere differentiable continuous Brownian path, parametrized by time. This is *a model* for a physical Brownian trajectory of an actual particle. Actual Brownian particles do not follow paths that are *precisely* of this nature.

If the signal to be analyzed is an image, then why not select a fixed but suitable *resolution* (or a subspace of signals corresponding to a selected resolution), and then do the computations there? Of course, the selection of a fixed "resolution" is dictated by practical concerns. That idea was key in turning computation of wavelet coefficients into iterated matrix algorithms. As the matrix operations get large, the computation is carried out in a variety of paths arising from big matrix products. Such paths have been studied in probability since Andrei N. Kolmogorov in the nineteen-thirties, but paths are perhaps better known in their continuous variants. Yet, what we know about the continuous case is the result of limit considerations arising from the discrete case. The dichotomy, continuous vs. discrete, is quite familiar to engineers. The industrial engineers typically work with huge volumes of numbers.

Numbers! — So why wavelets? Well, what matters to the industrial engineer isn't really the wavelets, but the fact that special wavelet functions serve as an efficient way to encode large data sets, I mean encode for computations. And the wavelet algorithms are computational. They work on numbers. Encoding numbers into pictures, images, or graphs of functions comes later; perhaps at the very end of the computation. But without the graphics, I doubt that we would understand any of this half as well as we do now. The same can be said for the many issues that relate to the crucial mathematical concept of self-similarity, as we know it from fractals, and more generally from recursive algorithms.

Prerequisites and cross-audience. I have used preliminary versions of this book in my courses. These courses fit the bill: "a second course in analysis", in one form or the other. Some of my courses were more traditional, and for math students, while others served a quite mixed audience, including students from engineering. At my university, there is a serious demand for interdisciplinary math courses; not only involving engineers, but also students from physics, statistics, CS, finance, and more. And I expect that this is a national trend. However, we found that many traditional math texts are too narrowly specialized and not well suited for cross-audiences. Or even well suited for a wider focus within math; for example being relevant to computational math, and to numerical analysis.

The students in my courses typically had some familiarity with function theory and with measures. But their background was always diverse. To accommodate diverse audiences (referring to the level of mathematical maturity, specialization, etc.), I included in my courses, and in the book, facts from a variety of topics.

How? Some of the exercises serve this very purpose, assimilation. For example, Exercises 1.9 through 1.12 have the flavor of *tutorials*; they let students pick up on some quite basic but central points from Fourier series, inner-product spaces, linear transformations, matrices (finite and infinite), and Hilbert space. Working the exercises is the best way for students to learn and to review fundamentals! In these multi-part exercises, the reader is then guided step by step through the issues; but as they are needed in the text. As other basic topics are needed later in the book, there are then other multi-part exercises which accomplish the same goal; e.g., Exercise 2.6 (integral operators), Exercises 2.7 through 2.10, and 9.8–9.9 (Brownian motion, including the fractional variant), Exercises 7.2 through 7.7 (matrix theory), and Exercises 7.8 through 7.10 (tensor product of Hilbert space, and product measure). An advantage of this approach is that students then see, and learn, these preliminaries precisely in the form in which they are used in the course.

When I was teaching mixed groups of students, the engineers in my class typically came from the following departments, EE (electrical engineers), electrical and computer engineering, communications, mechanical, industrial engineering, data mining, sensors—among others! Generally, the engineers looked for an agreeable mathematical presentation of central ideas from signal and image processing. ("Why then wavelets?" you may say. Reason: Wavelet *algorithms* share a lot in common with signal/image processing schemes! And they are *used* by engineers!)

Some of the engineers in my course worked at our University Hospital, a large teaching hospital. Their projects involve designing and improving technology in medical imaging.

I recently gave one of my thesis students (with a good background in math and engineering) the following assignment: "Work out the mathematics used in the processing of color images in a digital camera!" Reason: This is actually lovely algorithmic matrix theory, and it's based directly on wavelet algorithms.

Challenge! There is a huge difference in jargon used by engineers and by mathematicians (not to mention the other disciplines!); and often there's a call for some serious translation of technical lingo before you discover that the two groups are talking about the same thing. At times, my course would begin with first overcoming the cultural and the communication barriers; hence the system of interrelated *appendices* on polyphase matrices, and the *list of names and discoveries* in the back of the book.

Aim and scope. The aim of this book is to show how to use processes from probability, random walks on branches, and their path-space measures in the study of convergence questions from harmonic analysis, with particular emphasis on the infinite products that arise in the analysis of wavelets.

The focus of this book is by nature interdisciplinary, and it is motivated by some new mathematical trends that are still somewhat in a state of flux. We outline how they combine diverse areas from math and engineering in unexpected ways. As a result, we aim to address a diverse audience, perhaps unusually diverse, ranging from pure to applied (engineering and physics), from probability (random walk) to analysis (infinite products), from wavelets to fractals, from linear to nonlinear, from function theory to non-abelian operator algebras, from Lebesgue to Hausdorff measure, and from classical (Fourier) to modern (wavelets and more generally scale-similarity in time and space). This diversity presents us with a challenge, and we have taken pains in articulating the interconnections, describing a coherent unity, and in addressing the union (and not the intersection) of the various groups of readers who have an interest in this particular area of mathematics. We have aimed our exposition at the beginning graduate student level, having in mind both students and workers in a variety of fields, meeting at the crossroads where they merge into the subject at hand.

Self-similarity. The idea of self-similarity is common to the construction of wavelets, of various fractals, and of graph systems. These subjects may seem diverse, and they are often thought of as disparate. However we hope to outline how the notion of self-similarity runs through these subjects as a red thread connecting central themes: themes from harmonic analysis, from discrete mathematics, from probability, and from operator algebras. In addition, we list the two papers [**BoNe03**] and [**Nek04**] as sources that cover this from a somewhat different but related angle. Indeed, the concept of self-similarity has proved ubiquitous as well as fundamental in mathematics, and in diverse applications, perhaps because it serves to *renormalize* a rich variety of structures on nested families of scales (for example, similarity scales in time and/or in space). In wavelet theory, the scales may be represented in *resolutions* (a name deriving from optics) taking the form of nested systems of linear spaces. Similar such systems occur in fractal theory, and in dynamics. And in quantum field theory [**Nel73**], the self-similarity idea underlies the notion of "renormalization group", while in C^*-algebra theory [**Dav96**], it gives rise to representations of algebras on relations and generators, such as operator algebras on infinite particle systems [**PoSt70**], or the algebras of Cuntz, or Cuntz-Krieger. In fact, the pioneering paper [**Bat87**] early on stressed a fascinating connection between the renormalization group in quantum physics on the one hand, and a certain fundamental wavelet construction on the other.

To help the reader better see the bigger picture which we wish to present, we have included a system of interrelated appendices outlining a certain geometric approach to subband filtering which is both more operator-theoretic and more general than is usual. Our appendices build a bridge between two ways of doing wavelet/fractal subdivision algorithms, one based on quadrature conditions and their generalizations, and the other based on unitary operator functions and on representations of a certain C^*-algebra.

Recent developments in operator theory, approximation theory, orthogonal expansions, and wavelets demonstrate the need for a combination of techniques from analysis and probability. Since the relevant tools and techniques for these new applications are often interdisciplinary, they are not always readily available in the standard textbook literature. Or if they are, they are scattered over a number of older books or papers which are written for other purposes and aimed at other applications. The analysis problems that we focus on here are typically not presented together with their counterparts from probability theory: random walk, path space, infinite products, martingales, Kolmogorov extension techniques in their classical form; and in some of their modern non-commutative versions.

This book aims to fill an apparent gap in the literature, or at least to help bridge a gap. It gives a rigorous treatment of a number of convergence questions, and it also includes some new results. Our use of analysis in this book relies heavily on probabilistic tools, and the book offers a presentation of them.

New issues, new tools. This book is primarily about wavelets, but it takes a new direction in the subject: While it is about convergence issues, the questions come from applications with a twist that is different from the one that has dominated the recent literature.

The aim of this book is broader. We wish to show how some methods from probability, from operator algebras, and from random-walk theory can be used to prove theorems in analysis. The theme will be the use of processes from probability, and the use of random walk on branches and their path-space measures, in the study of convergence questions from harmonic analysis. Our emphasis is on the kinds of infinite products that arise in the analysis of wavelets, and more generally in the harmonic analysis of iterated function systems, and in dynamical systems.

Many modern wavelet constructions, from the past few years, are by necessity frequency-localized; and because of that fact, the convergence issues, and the tools that must be employed to resolve them, are completely different from the more traditional ones that were developed and tailored in the 1980's to time-localized functions, i.e., the wavelets that have the scaling function (the father function) and the wavelet function (the mother function) both of compact support. Those are the wavelets where multiresolution tools have been especially successful.

By contrast, the frequency-localized wavelets have compact support only in the Fourier-dual variable, and their resolution subspaces are typically not singly generated, so the more traditional multiresolution methods have fallen short, at least in the form in which they originally were given.

What this means is that some of the fundamental issues concerning pointwise convergence in the theory must be attacked with quite different tools: in this case, with tools from random walk, from probability, and from the theory of diffusions, processes, and martingales.

So the book is on the interface, and the applications, of probability, random walk, and path-space measures to convergence questions in harmonic analysis and in dynamics.

General theory. We consider a measurable space X with an endomorphism σ, mapping onto X, such that the inverse image of every point is of the same finite cardinality. The branches of the inverse are assigned probabilities by some positive function W, and we study the corresponding transition operator, also called the Perron–Frobenius–Ruelle operator $R = R_W$. By iteration, the branches determine a tree, and we study an associated random walk on this tree, and the transition measures indexed by the points in X.

While there is a considerable literature on this setup already, the various papers place some kind of regularity condition on W, or on the system, (X, σ), and the branches of the inverse. Our setting, assumptions and conclusions are in the measurable category. In contrast, when X is assumed to have a differentiable structure, and W is assumed Lipschitz, then, by Ruelle's theorem, there are solutions to the equations $\nu R = \nu$, $Rh = h$, and $\nu(h) = 1$, where ν is a Borel probability measure on X, and h is a nonnegative measurable function on X.

While the measure ν may not exist in the general measurable setting, we develop formulas for solutions h to the equation $Rh = h$, the so-called R-harmonic functions, and we give applications to the case when R is the wavelet transition operator defined from some measurable low-pass filter. In that setting, there are existence questions, and we show that the random-walk properties determine a variety of notions of wavelet orthogonality properties.

A word about the graphics and the illustrations. We owe much to the professional skill of Brian Treadway in programming and creating the computer generated renditions of the numerous figures in this book. They are an essential part of the exposition. Note that the Table of Contents in the front matter is followed by a List of our numbered Figures. This list includes the text for our figure captions. Readers are encouraged to preview the figures themselves to appreciate our use of trees, graphs, programming diagrams, subband-filtering schemes from engineering (signal/image processing), and a number of other visual tools and presentations. The author has himself found the figures, the graphs, and visualization of the algorithms exceptionally helpful in learning, in teaching and in discovering some of the material in this book; but more importantly, also in understanding the deeper connections in the subject.

The Mathematica program used in the production of Figure 7.5 is outlined by Brian Treadway in the narrative in the "References and remarks" section at the end of Chapter 7 inside the book. The two figures 7.4–7.5 (each figure spans two pages) serve to illustrate a particular aspect of the so-called pyramid algorithm for wavelets, i.e., the recursive algorithm which is used in among other things in the creation of wavelet packets; see, e.g., Figure 7.3. For this purpose, the algorithm is used in the two figures with two different initializations, the first is the simpler one, the Haar scaling function (see Figure 7.4), and it makes the dyadic subdivision especially transparent. The second initializes with Daubechies's scaling function (see Figures 7.5 and 7.16). The reason for the name "wavelet packet" (which is the subject of Chapter 7) is especially transparent in Figure 7.5. Notice especially two aspects of the progression of functions in the sequence of graphs inside the respective figures: In moving from one graph to the next, the numerical frequency appears to increase with each subdivision-step. But in addition, an enveloping shadow emerges in the progression through the graphs, and a shape of a wave with a lower frequency (i.e., longer wavelength) appears to "capture" and group the functions themselves into packets. We mention these figures already now, as the geometry of the steps that go into the diverse recursive algorithms are especially transparent to the naked eye in Figures 7.4–7.5. See also the sequence of figures 7.6–7.13 for the programming aspects of the same idea. Specifically, Figure 7.13 stresses the matrix steps that are indicated by the recursions.

We further stress that these figures play a central role in our presentation in this book: Some figures illustrate the kind of self-similarity in time and in space coordinates that is typical both in fractal analysis, and in the study of wavelets; while others illustrate decision trees; and yet others make clear the kinds of arrow-flow diagrams which are popular in building of recursive algorithms, and in programming more generally. See, e.g., [**Knu84**] as well as Knuth's monumental volumes [**Knu81**]. We mention Knuth's article [**Knu84**] as it emphasizes in a special case both the stochastic and the algorithmic features of the fundamental subdivision/filter algorithm.

Special features of the book. A main aim of the present book is to show how these ideas have proved fruitful in both the study of iterated function systems (IFS), see Chapter 4, and of wavelets, see Chapter 7. While the pyramid algorithm (in its diverse incarnations) is now typically identified with wavelets, it in fact has a long history in engineering, in information theory [**Sha49**], and in symbolic dynamics. The connection to signal and image processing from engineering was emphasized recently in our survey article [**Jor03**]; but see also [**DMSS03, Mal98, StNg96**] for much more detail. The connections to symbolic dynamics are manifold, but we wish to especially recommend the beautiful and inspiring invitation [**Rad99**] addressed to students, and authored recently by Charles Radin.

The book concludes with four separate items which we hope will help readers reconcile current terminology used simultaneously in math and in applications (especially in signal-processing engineering, and in physics—notably in optics!) These four elements are as follows: (i) a system of interrelated appendices, (ii) a list of comments for mathematicians on signal/image processing jargon, and (iii) a list of names (with historical comments) of mathematicians and scientists, both past and current era mathematicians/engineers/scientists who made pioneering contributions to the main ideas presented in the book.

Finally in item (iv) we include a *glossary* consisting of a list of terms occurring in the book in varied contexts of math, of probability, of engineering, and on occasion of physics. To reduce the apparent confusion created by the same concept having up to four different names, the glossary includes informal explanations spelling out the reasons behind the differences in current terminology from neighboring fields.

In item (i) we attempt to translate the various engineering terms and constructs into mathematical formulas. This is a continuation of a theme we started in a recent AMS Notices article [**Jor03**]. For (iii), we apologize for the subjective nature of our comments; and we readily acknowledge the difficulties in writing history of events that are rather modern on a mathematical scale. All three concluding additions to the book are motivated by the nature of the subject at hand, specifically by the many connections between ideas from signal/image processing and the more mathematical themes from wavelets, fractals, and dynamics.

Exercises. The exercises are essential, and they serve several pedagogical purposes: They are there to help students and users practice the fundamental concepts in the book, and to test his/her hand at computations, and at sketching functions or iterative schemes. But they also serve to expand horizons! We hope students will acquire a hands-on feeling for the various basis functions already discussed in the book; and more importantly, get started at building bases of his/her own. Some readers might even want to begin with the exercises, and then read the text as they work along in the exercises. Indeed, in designing the exercises, I have been inspired, at least in part by books with a philosophy like this: "Learn Hilbert space by doing problems!", e.g., [**Hal67**]. Or: "Operator algebras by example!", e.g., [**Dav96**]. Finally, several of our exercises have been put in to expand on themes inside the text, pointing the reader to new developments in the subject, and especially stressing links to neighboring subjects and to applications; pointing to connections between modern trends and classical subjects, highlighting an especially powerful (and beautiful) idea, or even suggesting unorthodox applications.

We have presented the material so that different readers can select the parts of it that are closest to his/her own interests; and in particular, it is not necessary to begin with Chapter 1. In fact, for some it might be better to begin with the Appendices, or with some of the special sections at the end:

"Comments on signal/image processing terminology", "Afterword"; or for some insight into the history of the subject, "List of names and discoveries". While some chapters build on earlier ones, non-sequential reading is possible, especially with the use of two extensive index lists, symbols and general. The following quote may be appropriate for interdisciplinary math books.

> *Det er ganske sandt, hvad Philosophien siger, at Livet maa forstaaes baglænds. Men derover glemmer man den anden Sætning, at det maa leves forlænds. Hvilken Sætning, jo meer den gjennemtænkes, netop ender med, at Livet i Timeligheden aldrig ret bliver forstaaeligt, netop fordi jeg intet Øieblik kan faae fuldelig Ro til at indtage Stillingen: baglænds.*[1]
>
> —Søren Kierkegaard

Iowa City,
October 2005 *Palle E. T. Jorgensen*

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[1] "It is really true what philosophy tells us, that life must be understood backwards. But with this, one forgets the second proposition, that it must be lived forwards. A proposition which, the more it is subjected to careful thought, the more it ends up concluding precisely that life at any given moment cannot really ever be fully understood; exactly because there is no single moment where time stops completely in order for me to take position [to do this]: going backwards." Often shortened to "Life can only be understood backwards; but it must be lived forwards" (*Livet skal forstaas baglæns, men leves forlæns*).

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