Physical Knots

Jonathan Simon *

November 2, 2000

preliminary version 13, 11/2/00, 4:00 pm
COPY JUST FOR TESTING DOWNLOAD
NOTES ARE NOT COMPLETE AND NOT READY FOR DISTRIBUTION

Abstract

What happens to knot theory when the knots, traditionally studied as purely one dimensional, completely flexible filaments, are given physical substance in the form of thickness, rigidity, or some kind of self-repulsion? Researchers have developed several measures of knot complexity, modeled on these kinds of physical "reality". We shall explore these ideas, see relations between different notions of complexity, and compare the "ideal" conformations of knots that arise. We also note that there are strong relations between these measures of complexity and behavior of actual knotted DNA molecules. Audience members will receive a genuine piece of rope and some easy-to-understand unsolved problems.

1 Introduction

Knots meet science in three different ways: At the most straightforward level, we can try to understand actual tangible objects that occupy space, have mass, etc. They may be large, as in Figures 1, 2, 3, 4, 6, 7, 8, 9, 10 , or microscopic, as in Figures 11 and 12.

*Department of Mathematics, University of Iowa, Iowa City, IA 52240, jsimon@math.uiowa.edu. Partial support provided by NSF Grant #DMS9706789 and the University of Iowa's Obermann Center for Advanced Studies

Next up in abstraction are the 1-dimensional knots that might occur as flow-lines in a fluid flow or other physical system (see talk by R. Ricca, also papers by H.K. Moffatt on knotted flux tubes [9], G. Buck on knotted n-body orbits [1], L. Faddeev and A. Niemi on knotted solitons [4], R. Ghrist and earlier work by J. Birman and R. Williams on knotted orbits in ODE flows).

There is a level even more subtle than flow-lines: the most abstract connection between knots and science is the phenomenon we might call "analogous patterns", where purely mathematical definitions and relationships in abstract knot theory are echoed by definitions and relationships in physics. (This is a connection developed by L. Kauffman.)

We began using the terms "physical knots" or "physical knot theory" in 1996, as the title of an AMS Special Session. One can see how the field has been developing by reviewing the lists of talks for that session[1] and the IMACS2000 session on Physical Knots [2]. The 1998 book *Ideal Knots* **??** provides an accessible entré, with expository articles by many of the people working in this area.

In this talk, as in the presentations on DNA and polymers by D. Sumners, A. Stasiak, and S. Whittington, we concentrate on knots made of real physical "stuff" that one can perceive and handle, and on mathematical models that seek to capture some of the physical properties.

2 "Strength" of knots

Here is a physical knots problem (actually a cluster of several) whose statement is immediately accessible to all of us and to our students. The problem appears to be well-understood in qualitative terms, by the engineers and others who encounter it in day to day applications; but I believe there is not yet an overall theory (nor will this talk provide one) to explain all that is observed and, in particular, that would give good quantitative predictions. We introduce the idea here in order to motivate some particular questions later, and also just because the phenomenon is so simple and intriguing.

People who enjoy fishing or sewing are familiar with the phenomenon that a string with a knot tied in it will break more readily than the same string without the knot. Some books of knots include the results of experiments on different knots, reporting the "strength of the knot" as the ratio

[1] http://at.yorku.ca/d/a/a/a/03.htm

[2] http://www.haverford.edu/math/ rmanning/imacs2000.html

[or percentage ratio]

$$\frac{\text{breaking strength of string with a knot tied in it}}{\text{breaking strength of same string with no knot}}\ .$$

This fraction appears to vary according to the type of knot. Why?

We can see in Figure 2 that the rope is bent where it emerges from the knot. Presumably this bending is one of the primary causes of weakening. There may also be an effect due to compression of the rope.

The strength also varies with the kind of string being used. One study[3] comparing different brands of fly-fishing line found that tying an overhand knot in one brand of line produced a small decrease in breaking strength, while doing the same thing to a different brand produced a much larger weakening. What geometric or physical properties of the fishing line could account for this? It appers that the line's diameter is at least one factor.

Because the knot-strength of a particular material varies with knot type, it is usually defined in terms of an overhand knot.

The knot-strength phenomenon is even recognized in international trade disputes (at least one, anyway). In 1994, the Canadian International Trade Tribunal decided [4] that U.S. companies were dumping certain kinds of twine, defined in terms of knot-strength. Here are excerpts from the report:

> The Canadian International Trade Tribunal, under the provisions of section 42 of the Special Import Measures Act, has conducted an inquiry following the issuance by the Deputy Minister of National Revenue for Customs and Excise of a preliminary determination of dumping dated December 23, 1993, and of a final determination of dumping dated March 23, 1994, respecting the importation into Canada of synthetic baler twine with a knot strength of 200 lbs or less, originating in or exported from the United States of America.
>
> ....
>
> Synthetic baler twine is used with agricultural baling equipment to bind bales of hay or straw. Baling equipment is designed specifically to produce either square or round bales. In the case

[3]Bill Nash, http://www.flyfishingreview.com/topics/archive/leadertest.html or http://hometown.aol.com/billsknots/ldrtst.htm

[4]http://www.tcce.gc.ca/dumping/Inquirie/Findings/nq93003e/nq93003e.htm

> of square balers, the twine is knotted. As square bales usually undergo considerable physical manipulation, the knot strength of the twine is an important consideration. In the case of round balers, the twine is merely wound around the bale a number of times, and the twine is not stressed by either knotting or the physical manipulation of the bale itself. Nevertheless, in round baling, the tensile strength of the twine is still an important consideration.
>
> ....
>
> It should be noted that knot strength and tensile strength are not interchangeable terms. Synthetic baler twine with a knot strength of 130 lbs may have a considerably higher tensile strength.

The question of knot-strength is also of interest to chemists, molecular biologists, and others working on understanding the physics, in particular breaking strength, of polymers.

In his talk at the 1996 AMS meeting in Iowa, E. Wasserman described computer simulations designed to analyze the effect of pulling tight a knot tied in a long chain molecule. A computationally intensive simulation of breaking knotted polyethylene, is reported in [10]. The chains were pulled until they broke, typically right at the point where the filament "enters" the knot. This led one of the authors to later say, "...it suggests that knots are topological objects that have universal properties that do not depend on their size"[6] [5].

Researchers have developed laboratory techniques to manipulate individual polymer molecules. In a 1999 experiment, polysaccharide molecules were attached at their ends and stretched until they broke [5]. Since the breaking strength was much less than the expected breaking strength of the polymer itself, the authors concluded that the break was happening at the points where the molecule was attached to the supports, rather than somewhere in the middle of the polymer chain. This led to a debate [11] between the authors and another group about whether the observed low breaking strength actually might be due to knotting in the long molecules.

[5] http://www.npaci.edu/envision/v16.2/biopolymers.html

3 Getting to the mathematics

The act of pulling a knot tight leads to interesting mathematical questions. Modulo the transition from open-rope knots to closed loops, if we tie an overhand knot (a common name for the simplest open-rope knot that we call the trefoil when we think of closed loops] in a rope and pull it tight, we always seem to get a figure that looks like Figures 9, 10, 15, 16, 17. The tight knot in a rope has something fundamental in common with conformations produced in very different settings: a living creature performing life functions, a curve given by by parametric equations, a curve (actually a polygon with 100 edges, so it looks like a smooth curve) found by computer simulation to maximize the ratio of diameter to length of the "virtual rope", and a curve (100 edge polygon again) obtained from the previous one by lowering a certain self-repelling energy function (see later section).

So the first tempting question is whether there exists, for each knot type, a universally optimal conformation. The two Figures 16, 17 teach us that different ways of measuring the complexity of knot conformations can lead to different kinds of "ideal" conformations. The book [12] exlores this theme in many directions. Despite its provocative title, that collection of papers really shows that there is no single "ideal". Nevertheless, it remains an interesting question of philosophy, and perhaps even mathematics, to decide why all the optimum conformations look so similar.

Given a particular measure of geometric complexity of knot conformations, we can ask basic mathematical questions:

- Does each knot type contain an optimal conformation? (Any geometric features of such an optimum conformation would be a new topological knot-type invariant.)

- Is the optimal conformation unique? Even if there is only one absolute minimizer, are there local minima? (So one might talk about the "energy spectrum" [9] of a knot type as a topological invariant.)

In the following sections, we discuss these and other questions in the context of two physically motivated ways of assigning numbers to knot conformations, attempting to measure how crumpled or crooked or otherwise complicated they are.

Figure 1: Rope Knot Loose

Figure 2: Rope Knot Pulled Tight

4 Thickness of Knots

How much rope does it take to make a knot? This question was posed to the author in 1985 by L. Siebenmann. The answer is still unknown.

We must first agree on a mathematical definition of “rope”.

5 Historical note

I learned last summer of a paper [7] published in 1976 that began exploring a number of ideas about mathematically modeling physical knots that many of us have subsequently rediscovered. It appears that those of us who sub-

Figure 3: Loose Knot Made of Mouse Cable

Figure 4: Tight Knot Made of Mouse Cable

Figure 5: Cut in different places and then tighten

Figure 6: Loose Knot Made of Chain

Figure 7: Tight Knot Made of Chain

Figure 8: Alternate Conformation of Tight Knot Made of Chain

Figure 9: Drawing of a hagfish.
http://www.zoology.ubc.ca/labs/biomaterials/slime.html

Figure 10: Actual hagfish knotting
http://oceanlink.island.net/oinfo/hagfish/hagfish.html

Figure 11: DNA Knot [13]

Figure 12: DNA Knot [3] [2]

Figure 13: Thick Tubes [8]

Figure 14: Doubly-Critical Self-Distance may be strictly larger than Singly-Critical Self-Distance

Figure 15: Different knots made from the same length and thickness of ”rope” have different overall/average sizes. This is why different DNA knots travel at different speeds in gel electrophoresis.