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Positive solutions to semilinear problems with coefficient that changes sign

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1. Introduction

In this paper we shall discuss the existence and nonexistence of a positive solution to the problem:

$$
\begin{aligned}
&-\Delta u = \lambda a(x) f(u), \quad x \in \Omega, \\
&u(x) = 0, \quad x \in \partial\Omega,
\end{aligned}
\tag{1}
$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary. We shall concentrate on the case when the coefficient $a(x)$ is allowed to change sign, which is of particular mathematical interest.

Due to its appearance in many mathematical models in physics, several special cases of Eq. (1) have been studied since the middle of 1800s. For example, it was used by Lord Kelvin [18] and Lane [13] to study the equilibrium configuration of mass in a spherical cloud of gas; this model was further studied by Fowler [8], leaving in the literature the Emden–Fowler equation which, in a generalized form, is still studied today, as seen in, for example, [2]. In nuclear physics, a form of (1) was proposed and used by Fermi [7] and by Thomas [17], and now it is widely known as the Thomas–Fermi equation. Both the Emden–Fowler equation and the Thomas–Fermi equation continue to be subjects of considerable interest in theoretical physics to this day, as a search on the Wide World Web will quickly convince the reader.

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In combustion theory Eq. (1) has also been used as a model, for example, in [1, 3, 5, 9, 15]. For a review of the work done up to 1975 in the ordinary differential equations setting, see [19]; for a review of the work done for the existence of positive solutions of semilinear elliptic equations, see [15].

Positive solutions are usually the ones of interest and because of the difficulties associated with proving the existence of such solutions using the techniques of nonlinear functional analysis, most of the recent work assumes nonnegativity of $a(x)$ and $f(u)$ in order to generate positive operators using the Green's function of $-\Delta$ on the region where the problem is posed.

Here, we are concerned with the case when the function $a(x)$ is allowed to change sign. In particular, we are interested in determining sufficient conditions on $a(x)$ to assure the existence of a positive solution for small values of λ, and to determine ranges of λ for which no positive solution exists. The main motivation for this work is the inherent mathematical difficulties it presents.

Finally, we note that the existing literature on Eq. (1) with the function $a(x)$ changing sign is much more limited. Among others we may mention [11, 16] which investigate (1) in different settings than ours.

2. The general case: existence

In this section, we shall tackle the general question of existence of a positive solution of (1).

Theorem 1. *Suppose that* $g\in L^\infty(\mathbb{R},\mathbb{R})$, *and* $a\in L^s(\Omega)$ *with* $s>1$. *Then for every* $\lambda\in\mathbb{R}$ *the Dirichlet problem*

$$\begin{aligned} -\Delta u &= \lambda a(x)g(u(x)), \quad x\in\Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2}$$

has a solution $u_\lambda\in W^{2,s}(\Omega)$.

Proof. For any $v\in L^s(\Omega)$ and any $\mu\in[0,1]$ let $T_\mu(v)\in W^{2,s}(\Omega)$ be the solution to the Dirichlet problem

$$\begin{aligned} -\Delta u &= \mu\lambda a(x)g(v(x)), \quad x\in\Omega, \\ u(x) &= 0, \quad x\in\partial\Omega. \end{aligned}$$

For some constant $K>0$, independent of v, μ, λ, we have

$$\|T_\mu v\|_{L^s}\le\|T_\mu v\|_{W^{2,s}}\le K|\lambda|\ \|a\|_{L^s}\|g\|_{L^\infty}. \tag{3}$$

Thus, taking $\rho>K|\lambda|\ \|a\|_{L^s}\|g\|_{L^\infty}$ we see that the compact mapping T_μ of $L^s(\Omega)$ into itself has no fixed point on the boundary of the open ball $B_\rho(0)$ centered at 0 and of radius ρ of $L^s(\Omega)$. Therefore,

$$d(I - T_\mu, B_\rho(0), 0) = d(I - T_0, B_\rho(0), 0) = d(I, B_\rho(0), 0) = 1, \quad 0 \le \mu \le 1.$$

Thus, T_1 has a fixed point inside $B_\rho(0)$ and this implies that the Dirichlet problem (2) has a solution $u_\lambda \in W^{2,s}(\Omega)$. □

Theorem 2. *Let $a \in L^s(\Omega)$, $s > \max(1, N/2)$, and suppose that there exists $\varepsilon > 0$ such that the solution of the Dirichlet problem*

$$-\Delta w = a^+(x) - (1+\varepsilon)a^-(x), \quad x \in \Omega,$$
$$w(x) = 0, \quad x \in \partial\Omega, \tag{4}$$

where $a^+(x) = \max(0, a(x))$, $a^-(x) = a^+(x) - a(x)$, $x \in \Omega$ is nonnegative inside Ω. Suppose further that $f \in C(\mathbb{R}, \mathbb{R})$, $f(0) > 0$. Then there exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$ the Dirichlet problem (1) has a solution which is positive inside Ω.

Proof. Fix a large number $M > 0$ and define $g : \mathbb{R} \to \mathbb{R}$ as follows:

$$g(\zeta) = f(0), \quad \zeta \le 0,$$
$$g(\zeta) = f(\zeta), \quad 0 < \zeta \le M,$$
$$g(\zeta) = f(M), \quad M < \zeta.$$

By Theorem 1, the Dirichlet problem (2) with g so defined has a solution $u_\lambda \in W^{2,s}(\Omega)$. Since $s > \max(1, N/2)$, by Sobolev's imbedding theorem, $u_\lambda \in C(\bar{\Omega})$.

Fix a number $\gamma \in (0, \varepsilon/(2+\varepsilon))$. The continuity of g implies that there is $\delta \in (0, M)$ such that

$$|\zeta| < \delta \Rightarrow g(0) - g(0)\gamma < g(\zeta) < g(0) + g(0)\gamma.$$

Again the fact that $s > \max(1, N/2)$, (3) and the Sobolev imbedding theorem imply that there exists $\lambda_0 > 0$ such that

$$\lambda \in (0, \lambda_0) \Rightarrow \|u_\lambda\|_{C(\bar{\Omega})} < \delta.$$

If we denote by $G : \Omega \times \Omega \to \mathbb{R}$ the Green's function for $-\Delta$ with homogeneous boundary condition, then it is well known that $G(x,\zeta) > 0$, $(x,\zeta) \in \Omega \times \Omega$, and, for $x \in \Omega$ and $\lambda \in (0, \lambda_0)$:

$$\begin{aligned} u_\lambda(x) &= \lambda \left\{ \int_\Omega G(x,\zeta) a^+(\zeta) g(u_\lambda(\zeta))\, d\zeta - \int_\Omega G(x,\zeta) a^-(\zeta) g(u_\lambda(\zeta))\, d\zeta \right\} \\ &> \lambda \int_\Omega G(x,\zeta) \{a^+(\zeta)[g(0) - g(0)\gamma] - a^-(\zeta)[g(0) + g(0)\gamma]\}\, d\zeta \\ &> \lambda f(0)(1-\gamma) \int_\Omega G(x,\zeta) \left[a^+(\zeta) - \frac{1+\gamma}{1-\gamma} a^-(\zeta) \right] d\zeta. \end{aligned}$$

From the hypothesis concerning problem (4), it follows that

$$u_\lambda(x) > \lambda f(0)(1-\gamma)\left[(1+\varepsilon) - \frac{1+\gamma}{1-\gamma}\right]\int_\Omega G(x,\zeta)a^-(\zeta)\,\mathrm{d}\zeta.$$

By our choice of γ the right-hand side is nonnegative and thus we have $u_\lambda(x) > 0$, $x \in \Omega$. □

Notes. Theorem 2 improves the result of [4] in a number of ways.
First: [4] considers only the radially symmetric case.
Second: it uses the method of upper and lower solutions and, as a consequence, it must require that f be nondecreasing.
Third: [4] requires (cf. (16) of [4]) that for some $\tau > 0$

$$\int_0^t r^{N-1}a^+(r)\,\mathrm{d}r \geq (1+\tau)\int_0^t r^{N-1}a^-(r)\,\mathrm{d}r, \quad 0 \leq t \leq v, \tag{5}$$

where v is the radius of the ball $\Omega = B_v(0)$.

This condition implies that the radially symmetric solution $u(x) = u(|x|)$ of (4) is positive inside the ball Ω. In fact, since a cannot be identically zero in Ω, if $a^-(r) = 0$, $0 \leq r \leq v$, then, on the one hand, (5) holds for any $\tau > 0$; on the other hand for any $\varepsilon > 0$ the solution of (4) is positive inside the ball by the maximum principle. So let us consider the case a^- not identically zero on $[0, v]$. Let $\omega(x) = \omega(|x|)$ satisfy:

$$-\Delta\omega = a^+(x) - (1+\tau/2)a^-(x), \quad |x| < v,$$
$$\omega(x) = 0, \quad |x| = v.$$

Then,

$$[\omega'(r)r^{N-1}]' = -r^{N-1}[a^+(r) - (1+\tau/2)a^-(r)], \quad 0 < r < v, \tag{6}$$
$$\omega(v) = 0,$$
$$\omega'(0) = 0.$$

Therefore, by (5)

$$\begin{aligned}\omega'(r)r^{N-1} &= -\int_0^r t^{N-1}a^+(t)\,\mathrm{d}t + (1+\tau/2)\int_0^r t^{N-1}a^-(t)\,\mathrm{d}t \\ &\leq -(\tau/2)\int_0^r t^{N-1}a^-(t)\,\mathrm{d}t, \quad 0 \leq r \leq v. \end{aligned}\tag{7}$$

Since a^- is not identically zero, there exists $r_0 \in (0, v)$ such that the right-hand side of (7) is negative on $[r_0, v]$. It follows that ω' is nonpositive on $[0, v]$ and negative on $[r_0, v]$. Since $\omega(v) = 0$ it follows that the solution ω is positive inside the ball. Thus, hypothesis (4) of [4] is stronger than our hypothesis concerning Eq. (4).

We end this section with a result that holds in our general setting and will be of considerable importance in what is to come.

Lemma 3. *Let $a \in L^s(\Omega)$, $s > \max(1, N/2)$, $a \neq 0$, and consider the problem*:

$$-\Delta\omega = a(x), \quad x \in \Omega$$

$$-\Delta\omega_1 = a(x)\omega, \quad x \in \Omega$$

$$\omega = \omega_1 = 0 \quad on\ \partial\Omega.$$

If $\partial\Omega$ satisfies Hopf's boundary lemma, then

$$\frac{\partial\omega_1}{\partial v} < 0$$

on $\partial\Omega$, where v is the unit outward normal vector.

Proof. Let $W(x) = \omega_1(x) - \frac{1}{2}\omega^2(x)$. Then,

$$\begin{aligned}\Delta W(x) &= \Delta\omega_1(x) - [\omega(x)\Delta\omega(x) + |\nabla\omega|^2(x)]\\ &= -|\nabla\omega|^2(x).\end{aligned}$$

It follows that W is a strict superharmonic function on Ω unless $\nabla\omega$ is identically zero, which would imply that ω is identically zero and thus a must be identically zero.

Thus, under our assumptions, it must be the case that $\Delta W(x) \leq 0$ and is not identically zero on Ω, and $W = 0$ on $\partial\Omega$.

Hopf's boundary lemma now yields

$$0 > \frac{\partial W}{\partial v} = \frac{\partial\omega_1}{\partial v} - \omega\frac{\partial\omega}{\partial v} = \frac{\partial\omega_1}{\partial v}. \qquad \square$$

3. Existence and nonexistence

Now, we will concentrate our focus in adding conditions which will assure us of much more detailed information concerning the existence or nonexistence of positive solutions for the problem under study.

The most used hypotheses in this section are listed below. In specific instances we will add to them.

(H_1) $f : \mathbb{R} \to \mathbb{R}$ is continuous with $f(0) > 0$.
(H_2) $a \in L^s(\Omega)$, $s > \max(1, N/2)$.
(H_3) Ω is a bounded domain in $\mathbb{R}^N$, $0 \in \Omega$, and $\partial\Omega \in C^1(\Omega)$.
(H_4)$_\varepsilon$ $\omega^\varepsilon \equiv \int_\Omega G(x,\zeta)(a^+(\zeta) - (1+\varepsilon)a^-(\zeta))\,\mathrm{d}\zeta > 0$, $x \in \Omega$.
(H_5) $A(t) = \int_0^t r^{N-1}a(r)\,\mathrm{d}r \geq 0$ and for some $t_0 \in (0,1)$, $A(t_0) > 0$.

A direct consequence of Theorem 2 is the following:

Theorem 4. *Under* (H_1)–(H_4)$_\varepsilon$ *for some $\varepsilon > 0$, there exists $\lambda_0 \in (0,1)$ such that problem* (1) *has at least one positive solution for $\lambda \in (0, \lambda_0)$.*

Now, we will give our first nonexistence theorem illustrating that $(H_4)_\varepsilon$ with $\varepsilon=0$, in general, does not guarantee the existence of a positive solution, which shows that Theorem 4 in its generalization is optimal.

Theorem 5. *Let f and a satisfy* (H_1)–$(\mathrm{H}_4)_\varepsilon$ *with $\varepsilon=0$. If f is bounded, $f'(0)<0$ and there exists $x_0\in\partial\Omega$ such that $(\partial/\partial v)\omega^0(x_0)=0$. Then (1) has no positive for all positive λ sufficiently small.*

Proof. Let

$$\omega^0(x)=\int_\Omega G(x,\zeta)a(\zeta)\,\mathrm{d}\zeta,$$

where G is the Green's function $-\Delta$ on Ω.

Given our assumptions, there exists $x_0\in\partial\Omega$

$$\omega^0(x)>0,\quad x\in\Omega,$$

$$\frac{\partial\omega^0}{\partial v}=0\quad\text{at } x_0.$$

Let $u=\lambda W$ be any solution of (1). Then,

$$-\Delta W(x)=a(x)f(\lambda W(x)). \tag{8}$$

Since f is bounded, $W\to W^0\equiv f(0)\omega^0$ in $W^{2,s}(\Omega)$ as $\lambda\to 0$. We have

$$\begin{aligned}W(x)&=\int_\Omega G(x,\zeta)a(\zeta)f(\lambda W(\zeta))\,\mathrm{d}\zeta\\&=\int_\Omega G(x,\zeta)a(\zeta)[f(0)+f'(0)\lambda W(\zeta)+o(W(\zeta))]\,\mathrm{d}\zeta\\&=f(0)\omega^0(x)+\lambda f^1(0)\int_\Omega G(x,\zeta)a(\zeta)W(\zeta)[1+o(1)]\,\mathrm{d}\zeta\\&=f(0)\omega^0(x)+\lambda f'(0)\omega_\lambda(x),\end{aligned}$$

where

$$\omega_\lambda(x)=\int_\Omega G(x,\zeta)a(\zeta)W(\zeta)[1+o(1)]\,\mathrm{d}\zeta. \tag{9}$$

As $\lambda\to 0^+$, (8) becomes $\Delta W_0+af(0)=0$, and hence

$$\omega_\lambda(x)\to\int_\Omega G(x\zeta)a(\zeta)\omega(\zeta)f(0)\,\mathrm{d}\zeta=f(0)\omega_1(x).$$

Here, we have

$$-\Delta\omega_1(x)=a(x)\omega(x)\quad\text{on }\Omega,$$

$$\frac{\partial\omega_1}{\partial v}<0\quad\text{on }\Omega.$$

Hence,

$$\frac{\partial \omega_\lambda}{\partial v} < 0 \tag{10}$$

for λ sufficiently small. For these values of λ we get

$$\frac{\partial W(x_0)}{\partial v} = f(0)\frac{\partial \omega(x_0)}{\partial v} + \lambda f'(0)\frac{\partial \omega_\lambda(x_0)}{\partial v} > 0$$

and this together with the fact that $W = 0$ on $\partial B_1(0)$ implies that W is negative near the boundary. □

Notes. It is very easy to construct $a(x)$ that satisfies the conditions of Theorem 5. We will just give a simple example here. Let $\Omega = B_1(0)$ and for $\rho \in (0,1)$ define

$$a(r) = \begin{cases} 1, & 0 \le r \le \rho, \\ -1, & \rho < r \le 1. \end{cases}$$

Then there exists a unique $\rho_0 \in (0,1)$ such that

$$\omega^0(r) \equiv \int_{B_1(0)} G(x,\zeta)a(\zeta)\,\mathrm{d}\zeta > 0$$

and $\omega^{0\prime}(1) = 0$.

The same argument implies that if $f'(0) > 0$ then W is positive for λ small and positive.

An immediate consequence is:

Theorem 6. *If* (H_1)–$(\mathrm{H}_4)_0$ *hold, and in addition* $f'(0) > 0$ *then there is* $\lambda_0 \in (0,1)$ *such that problem* (1) *has a positive solution for* $\lambda \in (0, \lambda_0)$.

Next, we shall give a set of nonexistence results for large λ, which are in spirit close to the nonexistence result in [14, Theorem 3.1].

Theorem 7. *Assume that the* $\Omega = B_1(0)$, $a(x) = a(r)$ *in* $B_1(0)$ *and* $a : [0,1] \to \mathbb{R}$ *is nonincreasing, that* $f : \mathbb{R} \to \mathbb{R}$ *is locally Lipschitz and positive and such that the non-decreasing function* g *with* $g(u) = \min_{s \ge u} f(s)$ *satisfies the strictly superlinear growth condition* $\int_0^\infty \mathrm{d}u/g(u) < \infty$ *and there exists* $\varepsilon > 0$ *such that* $a(x) > \varepsilon$ *in* $B_\varepsilon(0)$.

Then the problem

$$-\Delta u = \lambda a f(u) \quad \text{in } B_1(0),$$

$$u > 0 \quad \text{in } B_1(0),$$

$$u = 0 \quad \text{on } \partial B_1(0)$$

has no solution if

$$\lambda \ge \frac{2N}{\varepsilon^3}\int_0^\infty \frac{\mathrm{d}u}{g(u)}.$$

Proof. If u is a positive solution, then by [10], u is radially symmetric and so:

$$u''+\frac{N-1}{r}u'+\lambda a(r)f(u)=0.$$

In $B_\varepsilon(0)$ we then have:

$$r^{N-1}u'(r)+\lambda\int_0^r s^{N-1}a(s)f(u(s))\,\mathrm{d}s=0.$$

This implies that

$$r^{N-1}u'(r)+\lambda\varepsilon\int_0^r s^{N-1}g(u(s))\,\mathrm{d}s\leq 0.$$

Since $u'<0$ in $(0,\varepsilon)$, we have that

$$r^{N-1}u'(r)+\lambda\varepsilon g(u(r))\int_0^r s^{N-1}\,\mathrm{d}s\leq 0.$$

We get

$$\frac{u'(r)}{g(u)}+\frac{\lambda\varepsilon}{N}r\leq 0.$$

Thus,

$$\int_0^\varepsilon\frac{u'(r)}{g(u(r))}\,\mathrm{d}r+\int_0^\varepsilon\frac{\lambda\varepsilon}{N}r\,\mathrm{d}r\leq 0.$$

This shows that

$$\frac{\lambda\varepsilon^3}{2N}\leq\int_{u(\varepsilon)}^{u(0)}\frac{\mathrm{d}u}{g(u)}\leq\int_0^\infty\frac{\mathrm{d}u}{g(u)}.$$

This yields the desired result. □

Theorem 8. *If there exists $x_0\in\mathbb{R}^n$ and $\varepsilon>0$ such that $B_\varepsilon(x_0)\subseteq\Omega$, $a(x)\geq\varepsilon$ on $B_\varepsilon(x_0)$ and $f(u)\geq\varepsilon g(u)\geq 0$, where g is nondecreasing, convex, with $\int_0^\infty \mathrm{d}u/g(u)<\infty$, then the problem*

$$-\Delta u=\lambda a(x)f(u)\quad in\ B_\varepsilon(x_0),$$
$$u>0\quad in\ B_\varepsilon(x_0)$$

has no solution if

$$\lambda\geq\frac{2N}{\varepsilon^4}\int_0^\infty\frac{\mathrm{d}u}{g(u)}.$$

Proof. Let $\bar{u}(r)=\fint_{\partial B_r(x_0)}u(x)\,\mathrm{d}S_x$ for $0<r<\varepsilon$.

In $B_\varepsilon(x_0)$ we have

$$\int_{\partial B_r(x_0)} [\Delta u + af(u)] = 0$$

and hence

$$\int_{B_r(x_0)} [\Delta u + \lambda\varepsilon^2 g(u)] \leq 0$$

in $B_\varepsilon(x_0)$.

Jensen's inequality implies that

$$\Delta\,\bar{u} + \lambda\varepsilon^2 g(\,\bar{u}) \leq 0.$$

Similarly, we get

$$\lambda < \frac{2N}{\varepsilon^4}\int_0^\infty \frac{\mathrm{d}u}{g(u)}. \qquad \square$$

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