

Eigenfunction and harmonic function estimates in domains with horns and cusps

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1 Introduction

An interesting example in the paper of Davies and Simon [5] was that of a horn-shaped domain in \mathbb{R}^d . By horn-shaped we mean domains of the form $D = \{(x, y) : x > 0, \|y\| < f(x)\}$ with $f : [0, \infty) \rightarrow (0, \infty)$ a function tending to zero as x tends to infinity. Davies and Simon [5] established sufficiently sharp estimates on the first Dirichlet eigenfunction of Δ_d (d -dimensional Laplacian) on D to determine when the Dirichlet heat semigroup on D is intrinsically ultracontractive. This last property is important as one can provide bounds in that case for all the eigenfunctions, the heat kernel, and Green function in terms of the first eigenfunction. Thus, if one gets precise bounds on the first eigenfunction and the domain is intrinsically ultracontractive, then one has precise estimates on other important analytic quantities associated to the domain. Several works have appeared providing such bounds, Bañuelos [1],[2], Bañuelos and van den Berg [3], Bañuelos and Davis [4], Lindemann, Pang and Zhao [8].

In this paper we shall obtain pointwise bounds for positive harmonic functions vanishing

on the lateral side of horn-shaped domains in \mathbb{R}^d as well as for Dirichlet eigenfunctions in these domains. Our most general horn-shaped domain is as follows. Let $R : [0, \infty) \mapsto U(d)$, the unitary group, $S : [0, \infty) \mapsto \mathbb{R}^{d-1}$, $f : [0, \infty) \mapsto (0, \infty)$ and suppose all are continuous (more conditions will be added in due course) and take Ω to be a bounded domain in \mathbb{R}^{d-1} . Then our domains are of the form

$$D = \{(x, y) : x > 0, R(x)(y - S(x))/f(x) \in \Omega\}.$$

Throughout this paper we will always assume that

$$\lim_{x \rightarrow \infty} f(x) = 0 \text{ and } \partial\Omega \in C^{2,\alpha},$$

for some $\alpha > 0$. Suppose \tilde{w}_1 is the first Dirichlet eigenfunction of the Laplacian for Ω and $\tilde{\lambda}_1$ is its corresponding eigenvalue. What we establish is that whenever u is positive and harmonic in D , vanishing at infinity and on $\partial D \cap \{(x, y) : x > 0\}$ then $u(x, y)$ is bounded from above and below by constant multiples of $\exp\left\{-\sqrt{\tilde{\lambda}_1 - \epsilon} \int_0^x \frac{d\tau}{f(\tau)}\right\} \tilde{w}_1\left(\frac{R(x)(y-S(x))}{f(x)}\right)$ and $\exp\left\{-\sqrt{\tilde{\lambda}_1 + \epsilon} \int_0^x \frac{d\tau}{f(\tau)}\right\} \tilde{w}_1\left(\frac{R(x)(y-S(x))}{f(x)}\right)$, respectively. The same bound is established for the first Dirichlet eigenfunction of the Laplacian for D . In fact, we treat harmonic functions and eigenfunctions for $L = \Delta_d + \alpha(x, y)\frac{\partial}{\partial x} + \beta(x, y)\nabla_y + h(x, y)$ under suitable conditions on α , β and h . Similar domains were considered in Bañuelos and van den Berg [3]. They treated $L = \Delta_d$ on domains with less regularity than ours and consequently have less sharp bounds.

We obtain sharper bounds in the special case $\Omega = B_{d-1}(0, 1) \subset \mathbb{R}^{d-1}$ and $R(x) \equiv I$, $S(x) \equiv 0$. If $L = \Delta_d + h$ and $Lu = 0$ on D , $u > 0$ on D , $u(x, 0) \rightarrow 0$ as $x \rightarrow \infty$ and $u = 0$ on $\partial D \cap \{(x, y) : x > 0\}$ or u is the first Dirichlet eigenfunction for L on D , we show $u(x, y)$ is bounded within multiplicative constants of $f(x)^{\frac{2-d}{2}} \exp\left\{-\sqrt{\lambda_1} \int_0^x \frac{d\tau}{f(\tau)}\right\} w_1\left(\frac{\|y\|}{f(x)}\right)$ where w_1 and λ_1 are the first Dirichlet eigenfunction and eigenvalue for Δ_{d-1} on $B_{d-1}(0, 1)$. This

lower bound (when $h \equiv 0$ and $y = 0$) has been recently established by Lindeman, Pang and Zhao [8].

A brief word about the organization of this paper, Section 2 contains statements of our main results which are proved in Section 3. In Section 4 we apply the ideas of Davies and Simon [5] to obtain bounds on heat kernels and other eigenfunctions in terms of bounds on the first eigenfunction when the horn-shaped domain is intrinsically ultracontractive. We also apply the Kelvin transformation to obtain bounds on harmonic functions and eigenfunctions on domains with a cusp. Some of the results we need along the way are found in Section 5, the appendix.

We are indebted to Zhongxin Zhao for pointing out this problem to us.

2 Main Results

In this section we shall state our results and establish notation. Suppose $\Omega \subset \mathbb{R}^{d-1}$ is a bounded domain with $C^{2,\alpha}$ boundary and $f : [0, \infty) \rightarrow (0, \infty)$ is a $C^{2,\alpha}$ positive function for which $\lim_{x \rightarrow \infty} f(x) = 0$. We will make some additional assumptions on f below (see (2.2)). Let $R : [0, \infty) \rightarrow U(d)$ be C^2 and $S : [0, \infty) \rightarrow \mathbb{R}^{d-1}$ also C^2 . Define $z(x, y) = \frac{R(x)(y-S(x))}{f(x)}$ for $y = (y^1, \dots, y^{d-1})$ and finally put $D = \{(x, y) : x > 0, z(x, y) \in \Omega\}$. This is our “horn-shaped” domain.

Given Holder continuous maps $\alpha, h : D \rightarrow \mathbb{R}$, and $\beta : D \rightarrow \mathbb{R}^{d-1}$ define an operator L by

$$Lu = \Delta_d u + \alpha(x, y) \frac{\partial u}{\partial x} + \beta(x, y) \nabla_y u + h(x, y)u,$$

where Δ_d is the d -dimensional Laplacian. We shall study solutions of

$$(2.1) \quad \left\{ \begin{array}{l} Lu = 0, \text{ on } D, \\ u > 0, \text{ on } D, \\ u = 0, \text{ on } \partial D \cap \{(x, y) : x > 0\}, \\ \lim_{x \rightarrow \infty} u(x, y) = 0. \end{array} \right.$$

We shall assume

$$(2.2) \quad \left\{ \begin{array}{l} \text{(i)} \quad \lim_{x \rightarrow \infty} \{\|f(x)R'(x)\| + \|S'(x)\|\} = 0, \\ \text{(ii)} \quad \lim_{x \rightarrow \infty} \{f^2(x)\|R''(x)\| + f(x)\|S''(x)\| + |f'(x)|\|R'(x)\|\} = 0, \\ \text{(iii)} \quad \lim_{x \rightarrow \infty} \{f(x) \left(\sup_{y \in S(x)+f(x)R^{-1}(x)\Omega} |\alpha(x, y)| + \sup_{y \in S(x)+f(x)R^{-1}(x)\Omega} \|\beta(x, y)\| \right)\} = 0, \\ \text{(iv)} \quad \lim_{x \rightarrow \infty} \{|f'(x)| + f(x)|f''(x)|\} = 0, \\ \text{(v)} \quad \lim_{x \rightarrow \infty} \sup_{y \in S(x)+f(x)R^{-1}(x)\Omega} f^2(x)|h(x, y)| = 0. \end{array} \right.$$

The conditions (2.2) are satisfied if e.g. f, f', f'' all tend to zero at infinity and α, β, h are bounded and R, S are C_b^2 (bounded derivatives up to 2^{nd} order) functions with $\lim_{x \rightarrow \infty} \|S'(x)\| = 0$.

In what follows, \tilde{w}_1 and $\tilde{\lambda}_1$ will be the first Dirichlet eigenfunction and eigenvalue of Δ_{d-1} on Ω , respectively, with $\sup_{\Omega} \tilde{w}_1 = \tilde{w}_1(z_0) = 1$ for some $z_0 \in \Omega$.

Our first result is the following estimate.

Theorem 2.1. *Suppose u is a solution to $Lu = 0$ in D , $u > 0$ on D , $u = 0$ on $\partial D \cap \{x > 0\}$, $\lim_{x \rightarrow \infty} u(x, y) = 0$ with L satisfying (2.2). Then, for every $\delta \in (0, \tilde{\lambda}_1)$, there exist $x_0 > 0$ and constant C such that*

$$C^{-1} e^{-\sqrt{\tilde{\lambda}_1 + \delta} \int_0^x \frac{d\tau}{f(\tau)}} \tilde{w}_1(z) \leq u(x, y) \leq C e^{-\sqrt{\tilde{\lambda}_1 - \delta} \int_0^x \frac{d\tau}{f(\tau)}} \tilde{w}_1(z), \quad x > x_0, z \in \Omega.$$

Corollary 2.2. *Under the conditions of Theorem 2.1,*

$$\lim_{x \rightarrow \infty} \sup_{y \in S(x) + f(x)R(x)^{-1}\Omega} \frac{\log(u(x, y))}{\int_0^x \frac{d\tau}{f(\tau)}} = -\sqrt{\tilde{\lambda}_1}.$$

We now focus on the special case:

$$\Omega = B_{d-1}(0, 1) \subset \mathbb{R}^{d-1}$$

$$D = \{(x, y) : x > 0, \|y\| < f(x)\}$$

and

$$(2.3) \quad \left\{ \begin{array}{l} Lu = \Delta_d u + hu = 0, \text{ in } D \\ u > 0, \text{ in } D \\ u = 0, \text{ on } \partial D \cap \{(x, y) : x > 0\} \\ \lim_{x \rightarrow \infty} \sup_{|y| \leq f(x)} u(x, y) = 0. \end{array} \right.$$

We denote by w_1 and λ_1 the first Dirichlet eigenfunction and eigenvalue for Δ_{d-1} on $B_{d-1}(0, 1)$ with $\sup_{B_{d-1}(0,1)} w_1 = w_1(0) = 1$. We will make the following assumption on f .

$$(2.4) \quad \left\{ \begin{array}{l} \text{(i)} \quad \text{The condition (v) of assumption (2.2) is satisfied,} \\ \text{(ii)} \quad f \sup_{|y| \leq f(x)} |h(x, y)| \in L^1[0, \infty), \\ \text{(iii)} \quad \frac{f'^2}{f} \in L^1[0, \infty), \\ \text{(iv)} \quad f'' \in L^1[0, \infty), \\ \text{(v)} \quad \lim_{x \rightarrow \infty} f^2 \cdot f'''(x) = 0 \text{ and } f f''' \in L^1[0, \infty), \\ \text{(vi)} \quad \lim_{x \rightarrow \infty} f^3 \cdot f''''(x) = 0 \text{ and } f^2 f'''' \in L^1[0, \infty). \end{array} \right.$$

Note that the condition (iv) of assumption (2.2) can be easily derived from (2.4) (iv) and (v). Then we shall establish

Theorem 2.3. *If u solves (2.3) and f and h satisfy (2.4), then there is some positive constant C such that*

$$C^{-1}f(x)^{\frac{2-d}{2}} \exp \left\{ -\sqrt{\lambda_1} \int_0^x \frac{d\tau}{f(\tau)} \right\} w_1 \left(\frac{\|y\|}{f(x)} \right) \leq u(x, y) \leq C f(x)^{\frac{2-d}{2}} \exp \left\{ -\sqrt{\lambda_1} \int_0^x \frac{d\tau}{f(\tau)} \right\} w_1 \left(\frac{\|y\|}{f(x)} \right),$$

for $(x, y) \in D$, $x > 1$.

3 Proofs of Main Results

We will use the following test function in the proof of Theorem 2.1,

$$u(x, y) = \exp \left\{ -c \int_0^x \frac{d\tau}{f(\tau)} \right\} w(z),$$

where $z = z(x, y)$. Notice that if we denote the derivative of z with respect to \cdot by $D \cdot z$, then

$$(3.1) \quad \left\{ \begin{array}{l} D_x z = \frac{R'(y-S)}{f} - \frac{RS'}{f} - \frac{f'}{f} z \\ D_x^2 z = \frac{R''(y-S)}{f} - \frac{2R'S'}{f} - \frac{2f'R'(y-S)}{f^2} + \frac{2f'RS'}{f^2} - \frac{RS''}{f} \\ \quad \quad \quad + \frac{2(f')^2}{f^2} z - \frac{f''}{f} z, \\ \frac{\partial z^i}{\partial y^j} = \frac{R_j^i}{f} \\ \frac{\partial^2 z^i}{\partial y^{j^2}} = 0 \\ \sum_{k=1}^{d-1} \left(\frac{\partial z^i}{\partial y^k} \right) \left(\frac{\partial z^j}{\partial y^k} \right) = f^{-2} \delta^{ij}, \text{ since } RR^* = I_{d-1}. \end{array} \right.$$

Then, by (3.1) we have

$$(3.2) \quad Lu = f^{-2} \exp \left\{ -c \int_0^x \frac{d\tau}{f(\tau)} \right\} \{ \Delta_{d-1} w + c^2 w + D_z^2 w (f D_x z, f D_x z) + \langle \nabla_z w, f^2 D_x^2 z \rangle + \alpha f \langle \nabla_z w, f D_x z \rangle + \langle f \beta, \nabla_z w f D_y z \rangle + (cf' + f^2 h + cf \alpha) w \}.$$

Define

$$(3.3) \quad \begin{aligned} \tilde{\mathcal{A}}w = & D_z^2 w (f D_x z, f D_x z) + \langle \nabla_z w, f^2 D_x^2 z \rangle + \alpha f \langle \nabla w, f D_x z \rangle \\ & + \langle f \beta, \nabla w f D_y z \rangle + (c f' + f^2 h + c f \alpha) w \end{aligned}$$

so that

$$Lu = f^{-2} \exp \left\{ -c \int_0^x \frac{d\tau}{f(\tau)} \right\} \{ \Delta_{d-1} w + c^2 w + \tilde{\mathcal{A}}w \}.$$

Under (2.2) the operator

$$\mathcal{A} = \Delta_{d-1} + c^2 + \tilde{\mathcal{A}}$$

is a small perturbation of $\Delta_{d-1} + c^2$.

Denote by \vec{n} the unit outward pointing normal on $\partial\Omega$ and for $\xi \in \partial\Omega$ put $x(t, \xi) = \xi - t\vec{n}$, $t \geq 0$.

Since $\partial\Omega$ is compact and C^2 , there is an $\eta' > 0$ such that $x(t, \xi)$, $0 \leq t \leq \eta'$ satisfies $x(t, \xi) = x(s, \zeta)$ if and only if $t = s$ and $\xi = \zeta$. Also, given $\delta > 0$ small enough, there is an η'' such that

$$(3.4) \quad -(\tilde{w}_1(x(t, \xi)))' > \delta \quad \text{for } 0 \leq t < \eta''.$$

This follows readily from the Hopf maximum principle (Lemma 3.4 of [7]). Put $\eta = \eta' \wedge \eta''$, and take φ to be a smooth function, $\varphi : [0, \infty) \rightarrow [0, 1]$ with $\varphi(t) = 1$ for $0 \leq t \leq \eta/2$, $\varphi(t) = 0$ for $t > \eta$ and φ monotone decreasing on $[\eta/2, \eta]$. Define $\psi : \Omega \rightarrow [0, 1]$ by setting $\psi(y) = \varphi(t)$ if $y = x(t, \xi)$ for some $\xi \in \partial\Omega$, $0 < t < \eta$ and putting $\psi(y) = 0$ otherwise.

Set, for $a > 0$, $\Omega(a) = \{x(t, \xi) : \xi \in \partial\Omega, 0 < t < a\}$ and extend \vec{n} to denote the unit tangent field to the flow $x(\cdot, \xi)$ on $\Omega(\eta')$. Now given $\epsilon > 0$, let $\tilde{w}_1^{\pm\epsilon}$ and $\tilde{\lambda}_1^{\pm\epsilon}$ be the first Dirichlet eigenfunction and eigenvalue, respectively, for $\mathcal{A}^{\pm\epsilon} = \Delta_{d-1} \mp \epsilon \psi \vec{n} \cdot \nabla$ on Ω with $\sup_{\Omega} \tilde{w}_1^{\pm\epsilon} = 1$. By Schauder estimates (Theorem 6.6 of [7]), there is a C_1 independent of

ϵ such that for $\epsilon \in (0, 1)$, and $|u|_{2,0,\Omega} \equiv \sum_{j=0}^2 \sup_{\Omega} \sup_{|\beta|=j} |D^{\beta}u|$, $D^{\beta} = \partial_x^{\beta_1} \partial_{y_1}^{\beta_2} \dots \partial_{y_{d-1}}^{\beta_d}$,
 $|\beta| = \sum_1^d \beta_i$,

$$|\tilde{w}_1|_{2,0,\Omega} + |\tilde{w}_1^{\pm\epsilon}|_{2,0,\Omega} \leq C_1 \sup \left\{ \sup_{\Omega} \tilde{w}_1, \sup_{\Omega} \tilde{w}_1^{\pm\epsilon} \right\} = C_1 .$$

Then the uniqueness of the first eigenfunction implies that $\tilde{\lambda}_1^{\pm\epsilon} \rightarrow \tilde{\lambda}_1$ and $\sup_{\Omega} |\tilde{w}_1^{\pm\epsilon} - \tilde{w}_1| \rightarrow 0$ as $\epsilon \rightarrow 0$.

Applying $\mathcal{A}^{\pm\epsilon}$ to $\tilde{w}_1^{\pm\epsilon} - \tilde{w}_1$ yields

$$\mathcal{A}^{\pm\epsilon}(\tilde{w}_1^{\pm\epsilon} - \tilde{w}_1) = (\tilde{\lambda}_1 - \tilde{\lambda}_1^{\pm\epsilon})\tilde{w}_1^{\pm\epsilon} + \tilde{\lambda}_1(\tilde{w}_1^{\pm\epsilon} - \tilde{w}_1) \pm \epsilon\psi\vec{\eta}\nabla\tilde{w}_1$$

and so by Schauder estimates (Theorem 6.6 [7]),

$$|\tilde{w}_1^{\pm\epsilon} - \tilde{w}_1|_{2,0,\Omega} \leq C_2(|\tilde{\lambda}_1 - \tilde{\lambda}_1^{\pm\epsilon}| + \sup_{\Omega} |\tilde{w}_1^{\pm\epsilon} - \tilde{w}_1| + \epsilon)$$

which implies $|\tilde{w}_1^{\pm\epsilon} - \tilde{w}_1|_{2,0,\Omega} \rightarrow 0$.

From (2.2) and the above estimates, we have

$$\sup_{\Omega} |\tilde{\mathcal{A}}\tilde{w}_1^{\pm\epsilon}| = o(1) \quad \text{as } x \rightarrow \infty .$$

In addition, from (3.4), we deduce there is an $\epsilon_0(\delta/2)$ such that

$$(3.5) \quad -\psi\vec{n} \cdot \nabla\tilde{w}_1^{\pm\epsilon} \geq \delta/2 \quad \text{on } \Omega(\eta/2), \epsilon < \epsilon_0(\delta/2) .$$

$$(3.6) \quad -\psi\vec{n}\nabla\tilde{w}_1^{\pm\epsilon} \geq 0 \quad \text{on } \Omega, \epsilon < \epsilon_0(\delta/2) .$$

Let $C_3 = \inf_{0 < \epsilon < \epsilon_0(\delta/2)} \inf_{\Omega \setminus \Omega(\eta/2)} \tilde{w}_1^{\pm\epsilon}$ and notice that $1 > C_3 > 0$ by Harnack's inequality.

From the fact that $\tilde{\lambda}_1^{\pm\epsilon} \rightarrow \tilde{\lambda}_1$, we see there is an $\epsilon_1(\delta, C_4)$ such that

$$(3.7) \quad |\tilde{\lambda}_1^{\pm\epsilon} - \tilde{\lambda}_1| < \delta/2, \quad 0 < \epsilon < \epsilon_1(\delta, C_4) .$$

Finally fix $\epsilon = \epsilon_0(\delta) \wedge \epsilon_1(\delta, C_4) \wedge 1$. Then there is an $x_0(\delta, \epsilon)$ such that

$$(3.8) \quad \sup_{\Omega} |\tilde{\mathcal{A}}\tilde{w}_1^{\pm\epsilon}| \leq C_4\epsilon\delta/2, \quad x > x_0(\delta, \epsilon) .$$

We can now prove

Proposition 3.1. *There exist positive ϵ, δ, x_0 such that*

$$u_-(x, y) = \exp \left\{ -\sqrt{\tilde{\lambda}_1 + \delta} \int_0^x \frac{d\tau}{f(\tau)} \right\} \tilde{w}_1^{-\epsilon}(z)$$

and

$$u_+(x, y) = \exp \left\{ -\sqrt{\tilde{\lambda}_1 - \delta} \int_0^x \frac{d\tau}{f(\tau)} \right\} \tilde{w}_1^{+\epsilon}(z)$$

are sub- and super-solutions, respectively, for L on $D \cap \{(x, y) : x > x_0\}$.

Proof. Take ϵ, δ, x_0 as in the discussion preceding the Proposition. We give the proof for u_- , that for u_+ being entirely analogous. Now

$$Lu_- = f^{-2} \exp \left\{ -\sqrt{\tilde{\lambda}_1 + \delta} \int_0^x \frac{d\tau}{f(\tau)} \right\} \mathcal{A}\tilde{w}_1^{-\epsilon}(z),$$

with

$$\mathcal{A}\tilde{w}_1^{-\epsilon}(z) = (\tilde{\lambda}_1 + \delta - \tilde{\lambda}_1^{-\epsilon})\tilde{w}_1^{-\epsilon}(z) - \epsilon\psi\vec{n} \cdot \nabla\tilde{w}_1^{-\epsilon}(z) + \tilde{\mathcal{A}}\tilde{w}_1^{-\epsilon}(z).$$

Now, on $\Omega(\eta/2)$, we have by (3.5), (3.7) and (3.8) that

$$-\epsilon\psi\vec{n} \cdot \nabla\tilde{w}_1^{-\epsilon} + \tilde{\mathcal{A}}\tilde{w}_1^{-\epsilon} \geq 0, x > x_0$$

and

$$(\tilde{\lambda}_1 + \delta - \tilde{\lambda}_1^{-\epsilon})\tilde{w}_1^{-\epsilon} \geq 0.$$

On $\Omega \setminus \Omega(\eta/2)$ we have by (3.6), (3.7) and (3.8) that

$$(\tilde{\lambda}_1 + \delta - \tilde{\lambda}_1^{-\epsilon})\tilde{w}_1^{-\epsilon} + \tilde{\mathcal{A}}\tilde{w}_1^{-\epsilon} \geq 0, x > x_0$$

and

$$-\epsilon\psi\vec{n} \cdot \nabla\tilde{w}_1^{-\epsilon} \geq 0.$$

Thus $\mathcal{A}\tilde{w}_1^{-\epsilon} \geq 0$ and u_- is a sub-solution as claimed. The proof that u_+ is a super-solution is entirely analogous □

Finally, we note that

Lemma 3.2. *There exist constants ϵ_0, C , so that for all $\epsilon < \epsilon_0$*

$$C^{-1}\tilde{w}_1^{\pm\epsilon}(z) \leq \tilde{w}_1(z) \leq C\tilde{w}_1^{\pm\epsilon}(z), \quad z \in \Omega.$$

Proof. The Hopf maximum principle (Lemma 3.4 [7]) gives a $\gamma > 0$ so that the inequality holds in $\Omega(\gamma)$. For $z \in \Omega \setminus \Omega(\gamma)$ it follows from Harnack's inequality. \square

Proof of Theorem 2.1. Combine Proposition 3.1, Lemma 3.2 and the maximum principle. \square

Now consider the special case $\Omega = B(0, 1) \subset \mathbb{R}^{d-1}$, $\alpha = 0, \beta = 0$, $R(x) \equiv I$ and $S(x) \equiv 0$.

Then $w_1(z) = w_1(y/f(x))$ is a function of $t = \|y\|/f(x)$ and (3.2) becomes

$$\begin{aligned} Lu = f^{-2} \exp \left\{ -c \int_0^x \frac{d\tau}{f(\tau)} \right\} & \left\{ w''(t) + \frac{1}{1+t^2 f'^2} \left(\frac{d-2}{t} + t(2cf' + 2f'^2 - f'') \right) w'(t) \right. \\ & \left. + \frac{1}{1+t^2 f'^2} (c^2 + cf' + f^2 h) w(t) \right\} \end{aligned}$$

We will now improve the bound of Theorem 2.1 by proving Theorem 2.3 under additional assumptions on f . However, Theorem 2.1 will be valid in this setting if the following condition is satisfied

$$\lim_{x \rightarrow \infty} \{ |f'(x)| + f'^2(x) + f(x)|f''(x)| + f^2(x) \sup_{|y| < f(x)} |h(x, y)| \} = 0.$$

We also observe that a moving plane argument shows that (please see Lemma 5.1 in the appendix for details) if $h(x, y) = h(x, \|y\|)$ with h nonincreasing in $\|y\|$ then

$$u(x, 0) = \max_{|y| < f(x)} u(x, y) \quad \text{if } u(0, y) = u(0, \|y\|) \text{ is also nonincreasing in } \|y\|.$$

Note both $\{f, h^+(x) = \sup_{\|y\| \leq f(x)} h(x, y)\}$ and $\{f, h^-(x) = \inf_{\|y\| \leq f(x)} h(x, y)\}$ satisfy (2.4) when $\{f, h\}$ does. Let $x_1 > 0$ be given by Lemma 5.3 (see Appendix). By the Hopf

boundary lemma (Lemma 3.4 of [7]) there exist $0 < c_1 < c_2$ such that

$$c_1 w_1\left(\frac{\|y\|}{f(x_1)}\right) \leq u(x_1, y) \leq c_2 w_1\left(\frac{\|y\|}{f(x_1)}\right).$$

And it is clear that both $\{f, h^+\}$ with $c_2 w_1(\frac{\|y\|}{f(x)})$ and $\{f, h^-\}$ with $c_1 w_1(\frac{\|y\|}{f(x)})$ satisfy Lemma 5.3. Let u^+ and u^- be the solutions of (5.3) for boundary data $c_2 w_1(\frac{\|y\|}{f(x)})$ and $c_1 w_1(\frac{\|y\|}{f(x)})$, respectively, on \tilde{D}_{x_1} . Then Lemma 5.2 implies that

$$u \leq u \leq u^+ \text{ on } D \cap \{(x, y) : x > x_1\}.$$

Furthermore it is clear that Lemma 5.1 applies to both u^+ and u^- . Hence it is sufficient to prove Theorem 2.3 for u^+ and u^- with $\{f, h^+\}$ and $\{f, h^-\}$, respectively. Therefore from now on in this section we shall assume that u solves (2.3) with

$$(3.9) \quad u(x, y) = u(x, r), \quad h(x, y) = h(x), \quad r = \|y\| \quad \text{and } u \text{ is decreasing in } r.$$

The proof relies on the following change of variables,

$$(3.10) \quad t = \frac{r}{f(x)} \quad (t \in [0, 1)), \quad s = \int_0^x \frac{d\tau}{f(\tau)} + \frac{1}{2} t^2 f'(x).$$

Given s and t , this defines $x = x(s, t)$ and for future use we record the formulas

$$(3.11) \quad \frac{\partial x}{\partial s}(s, t) = \frac{f}{1 + \frac{1}{2} t^2 f f''}$$

$$(3.12) \quad \frac{\partial x}{\partial t}(s, t) = \frac{-t f f'}{1 + \frac{1}{2} t^2 f f''}$$

$$(3.13) \quad \frac{\partial^2 x}{\partial t^2}(s, t) = \frac{-f f' - t(f'^2 + f'')x_t}{1 + \frac{1}{2} t^2 f f''} + \frac{t f f'(t f f'' + \frac{1}{2} t^2 (f' f'' + f f'''))x_t}{(1 + \frac{1}{2} t^2 f f'')^2}$$

$$(3.14) \quad \frac{\partial^2 x}{\partial s \partial t}(s, t) = \frac{f' x_t}{1 + \frac{1}{2} t^2 f f''} - \frac{f(t f f'' + \frac{1}{2} t^2 (f' f'' + f f'''))x_t}{(1 + \frac{1}{2} t^2 f f'')^2}.$$

From now on, f, f', f'' , etc., and h will be evaluated at $x(s, t)$. Then if u is a solution of (2.3), write

$$u(x, y) = u(x, r) = f(x)^{\frac{2-d}{2}} v(s, t).$$

Then, (2.3) forces v to satisfy

$$\begin{aligned}
\mathcal{A}v(s, t) &\equiv (1 - t^2 f'(x)^2 + t^4 f'(x)^4 + t^2 f(x) f''(x) - t^4 f(x) f'(x)^2 f''(x)) \\
&\quad + \frac{1}{4} t^4 f(x)^2 f''(x)^2 \frac{\partial^2 v}{\partial s^2}(s, t) + (1 + t^2 f'(x)^2) \frac{\partial^2 v}{\partial t^2}(s, t) \\
&\quad + \left(\frac{d-2}{t} + dt f'(x)^2 - t f(x) f''(x) \right) \frac{\partial v}{\partial t}(s, t) \\
&\quad + (2t^3 f'(x)^3 - t^3 f(x) f'(x) f''(x)) \frac{\partial^2 v}{\partial s \partial t}(s, t) \\
(3.15) \quad &\quad + (t^2 f'(x)^3 + dt^2 f'(x)^3 - 2t^2 f(x) f'(x) f''(x)) \\
&\quad - \frac{1}{2} dt^2 f(x) f'(x) f''(x) + \frac{1}{2} t^2 f(x)^2 f^{(3)}(x) \frac{\partial v}{\partial s}(s, t) \\
&\quad + (f(x)^2 h(x) + \frac{d}{2} \left(\frac{d}{2} - 1 \right) f'(x)^2 + \left(1 - \frac{d}{2} \right) f(x) f''(x)) v(s, t) \\
&= 0.
\end{aligned}$$

After dividing everything by the coefficient of $\frac{\partial^2 v}{\partial s^2}$, this equation is written in the following suggestive manner.

$$\begin{aligned}
\mathcal{A}v(s, t) &\equiv \frac{\partial^2 v}{\partial s^2}(s, t) + \frac{\partial^2 v}{\partial t^2}(s, t) + \frac{d-2}{t} \frac{\partial v}{\partial t}(s, t) \\
(3.16) \quad &\quad + \tilde{c}_1(s, t) \frac{\partial^2 v}{\partial t^2}(s, t) + \tilde{c}_2(s, t) \frac{\partial^2 v}{\partial s \partial t}(s, t) + \tilde{c}_3(s, t) \frac{\partial v}{\partial t}(s, t) \\
&\quad + \tilde{c}_4(s, t) \frac{\partial v}{\partial s}(s, t) + \tilde{c}_5(s, t) v(s, t) = 0
\end{aligned}$$

with

$$(3.17) \quad k(s, t) = f f'' - f'^2,$$

$$(3.18) \quad \ell(s, t) = t^2 k(1 - t^2 f'^2) + \frac{1}{4} t^4 f^2 f''^2,$$

$$(3.19) \quad \tilde{c}_1(s, t) = \frac{t^2 f'^2 - \ell}{1 + \ell},$$

$$(3.20) \quad \tilde{c}_2(s, t) = \frac{t^3(2f'^3 - f f' f'')}{1 + \ell},$$

$$(3.21) \quad \tilde{c}_3(s, t) = -\frac{t(df'^2 - f f'') - (d-2)t^{-1}\ell}{1 + \ell},$$

$$(3.22) \quad \tilde{c}_4(s, t) = \frac{(d+1)t^2 f'^3 - \frac{1}{2}(d+4)t^2 f f' f'' + \frac{1}{2}t^2 f^2 f'''}{1 + \ell},$$

$$(3.23) \quad \tilde{c}_5(s, t) = \frac{f^2 h + \frac{1}{4}d(d-2)f'^2 - \frac{1}{2}(d-2)ff''}{1 + \ell}.$$

Assumption (2.4) insures that for s_0 large enough $|\ell(s, t)| \leq \frac{1}{2}$ for $s \geq s_0$, $0 \leq t < 1$. In fact, $\lim_{s \rightarrow \infty} \sup_{|t| < 1} \ell(s, t) = 0$. Thus $\lim_{s \rightarrow \infty} \sup_{0 \leq t \leq 1} |\tilde{c}_i(s, t)| = 0$ for $i = 1, 2, 3, 4, 5$.

If the coefficients $\tilde{c}_i(s_i t)$ were identically zero then we would be considering $v = \tilde{v}$ with $\tilde{\mathcal{A}}\tilde{v} = \frac{\partial^2 v^2}{\partial s^2} + \frac{\partial^2 \tilde{v}}{\partial t^2} + \frac{d-2}{t} \frac{\partial \tilde{v}}{\partial t} = 0$. A solution to the latter equation suited to the boundary conditions in (2.3) is $\tilde{v}(s, t) = e^{-\sqrt{\lambda_1} s} w_1(t)$. This motivates the bounds in Theorem 2.3. Our strategy to obtain the estimate in Theorem 2.3 when the $\tilde{c}_i \not\equiv 0$ is to define

$$\varphi_1(s) = \int_0^1 w_1(t) t^{d-2} v(s, t) dt$$

and first compare $v(s, t)$ to $\varphi_1(s)w_1(t)$. Then we obtain estimates on φ_1 from examining the differential equation it must satisfy because of (3.16).

We have the following

Proposition 3.3. *There is an $N_1 > 1$ such that*

$$N_1^{-1} \sup_{0 \leq t \leq 1} v(s, t) \leq \varphi_1(s) \leq N_1 \sup_{0 \leq t \leq 1} v(s, t).$$

Proof. The right-hand inequality is automatic. The left-hand inequality follows from Harnack's inequality and the fact that $u(x, r)$ is decreasing in r . \square

Proposition 3.4. *There is an $N_2 \geq N_1$, $s_0 > 0$, such that for $s > s_0$,*

$$N_2^{-1} \varphi_1(s) w_1(t) \leq v(s, t) \leq N_2 \varphi_1(s) w_1(t).$$

Proof. First select $s_0 = \max\{1, \int_0^{x_1} \frac{d\tau}{f(\tau)} + \frac{1}{2}|f'(x_1)|\}$ where x_1 is given in Lemma 5.3, we shall make s_0 larger for the left-hand inequality.

By Schauder estimates (recall the norms from Section 2),

$$|v|_{2,0;[s-1/2,s+1/2] \times [0,1]} \leq C |v|_{0,0;([s-1,s+1] \times [0,1])}$$

the right-hand side is bounded by $CN_1\varphi_1(s)$ by Harnack's inequality, (5.4), the fact that $u(x, r)$ is decreasing in r and Proposition 3.3. Since $v(s, 1) = 0$, $v(s, t) \leq C_3(1-t)\varphi_1(s)$, for some $C_3 > 0$. But $w_1(t) \geq c_4^{-1}(1-t)$ which implies the right-hand side.

For the lower bound, we use a well-known technique due to Hopf. By Proposition 3.3, and Harnack's inequality, there is a positive constant c_5 so that

$$(3.24) \quad c_5\varphi(s) \leq v(s, t), \quad 0 \leq t \leq 3/4.$$

Define $\xi(t) = e^{-\alpha t^2} - e^{-\alpha}$. Then

$$\begin{aligned} \mathcal{A}\xi(t) &= e^{-\alpha t^2} [2\alpha(2\alpha - 1)(1 + \tilde{c}_1(s, t)) - 2\alpha((d - 2) + t\tilde{c}_3(s, t))] \\ &\quad + \tilde{c}_5(s, t)\xi(t) \\ &\leq e^{-\alpha t^2} [2\alpha(2\alpha - 1)(1 + \tilde{c}_1(s, t)) - 2\alpha((d - 2) + t\tilde{c}_3(s, t)) - \tilde{c}_5^-(s, t)]. \end{aligned}$$

Since $\tilde{c}_1 = o(1)$, $\tilde{c}_3 = o(1)$ and $\tilde{c}_5^- = o(1)$ as $s \rightarrow \infty$, if s_0 is large and $s \geq s_0$, we can select α large enough so that

$$(3.25) \quad \mathcal{A}\xi(t) \geq 0, \quad 3/4 \leq t \leq 1.$$

By (2.24), (2.25) and the maximum principle; there is a constant c_6 such that

$$c_6\varphi(s)\xi(t) \leq v(s, t), \quad 3/4 \leq t < 1.$$

But there is a $c_7 > 0$ so that

$$\xi(t) \geq c_7(1-t), \quad 3/4 \leq t < 1.$$

Since $(1-t) \geq c_8w(t)$, for some $c_8 > 0$ and $3/4 \leq t < 1$, the proof is complete. \square

We now multiply both sides of (3.16) by $w_1(t)t^{d-2}$ and integrate from $t = 0$ to $t = 1$.

The resulting equation, $\int_0^1 \mathcal{A}v(s, t)w_1(t)t^{d-2}dt = 0$, after integrations by parts, becomes

$$(3.26) \quad \left\{ \begin{array}{l} \varphi_1''(s) - \lambda_1\varphi_1(s) + \int_0^1 \frac{d^2}{dt^2}(\tilde{c}_1(s, t)w_1(t)t^{d-2})v(s, t)dt \\ - \int_0^1 \frac{d}{dt}(\tilde{c}_2(s, t)w_1(t)t^{d-2})\frac{\partial v}{\partial s}(s, t)dt - \int_0^1 \frac{d}{dt}(\tilde{c}_3(s, t)w_1(t)t^{d-2})v(s, t)dt \\ + \int_0^1 \tilde{c}_4(s, t)w_1(t)t^{d-2}\frac{\partial v}{\partial s}(s, t)dt + \int_0^1 \tilde{c}_5(s, t)v(s, t)w_1(t)t^{d-2}dt = 0 . \end{array} \right.$$

Introducing

$$(3.27) \quad \left\{ \begin{array}{l} g_1(s, t) = \frac{d^2}{dt^2}(\tilde{c}_1(s, t)w_1(t)t^{d-2}) \\ g_2(s, t) = -\frac{d}{dt}(\tilde{c}_2(s, t)w_1(t)t^{d-2}) \\ g_3(s, t) = -\frac{d}{dt}(\tilde{c}_3(s, t)w_1(t)t^{d-2}) \\ g_4(s, t) = \tilde{c}_4(s, t)w_1(t)t^{d-2} \\ g_5(s, t) = \tilde{c}_5(s, t)w_1(t)t^{d-2} . \end{array} \right.$$

Then (3.26) can be written

$$(3.28) \quad \left\{ \begin{array}{l} \varphi_1''(s) - \lambda_1\varphi_1(s) + \int_0^1 (g_1(s, t) + g_3(s, t) + g_5(s, t))v(s, t)dt \\ + \int_0^1 (g_2(s, t) + g_4(s, t))\frac{\partial v}{\partial s}(s, t)dt = 0 . \end{array} \right.$$

Lemma 3.5. *Under assumption (2.4)*

$$\varphi_1'(s) < 0 \text{ at } \infty .$$

Proof. From Proposition 3.4 and (5.4) we have for some C that

$$(3.29) \quad C^{-1}e^{-\frac{3\sqrt{\lambda_1}}{2}(s-s_1)}\varphi_1(s_1)w_1(t) \leq v(s, t) \leq Ce^{-\frac{\sqrt{\lambda_1}}{2}(s-s_1)}\varphi_1(s_1)w_1(t) \quad s \geq s_1 \geq s_0 ,$$

and then

$$(3.30) \quad C^{-1}e^{-\frac{3\sqrt{\lambda_1}}{2}(s-s_1)}\varphi_1(s_1) \leq \varphi_1(s) \leq Ce^{-\frac{\sqrt{\lambda_1}}{2}(s-s_1)}\varphi_1(s_1) \quad s \geq s_1 \geq s_0 .$$

Next integrate (3.28) from s to ∞ , obtaining

$$(3.31) \quad \left\{ \begin{array}{l} \varphi_1'(s) = -\lambda_1 \int_s^\infty \varphi_1(\sigma)d\sigma + \int_s^\infty \int_0^1 (g_1(\sigma, t) + g_3(\sigma, t) + g_5(\sigma, t))v(\sigma, t)dtd\sigma \\ + \int_s^\infty \int_0^1 (g_2(\sigma, t) + g_4(\sigma, t))\frac{\partial v}{\partial \sigma}(\sigma, t)dt d\sigma . \end{array} \right.$$

Integration by parts gives

$$(3.32) \quad \left\{ \begin{array}{l} \varphi_1'(s) = -\lambda_1 \int_s^\infty \varphi_1(\sigma) d\sigma + \int_s^\infty \int_0^1 (g_1(\sigma, t) + g_3(\sigma, t) + g_5(\sigma, t)) v(\sigma, t) dt d\sigma \\ - \int_0^1 (g_2(s, t) + g_4(s, t)) v(s, t) dt - \int_s^\infty \int_0^1 \frac{\partial(g_2+g_4)}{\partial\sigma}(\sigma, t) v(\sigma, t) dt d\sigma . \end{array} \right.$$

By (3.4) and (3.30) we obtain that for $s \geq s_0$

$$(3.33) \quad \left\{ \begin{array}{l} \varphi_1'(s) \leq -\lambda_1 \int_s^\infty C^{-1} e^{-\frac{3\sqrt{\lambda_1}}{2}(\sigma-s)} \varphi_1(s) d\sigma \\ + \int_s^\infty \int_0^1 |g_1(\sigma, t) + g_3(\sigma, t) + g_5(\sigma, t)| C e^{-\frac{\sqrt{\lambda_1}}{2}(\sigma-s)} \varphi_1(s) w_1(t) dt d\sigma \\ + \int_0^1 |g_2(s, t) + g_4(s, t)| C e^{-\frac{\sqrt{\lambda_1}}{2}(s-s)} \varphi_1(s) w_1(t) dt \\ + \int_s^\infty \int_0^1 \left| \frac{\partial(g_2+g_4)}{\partial\sigma}(\sigma, t) \right| C e^{-\frac{\sqrt{\lambda_1}}{2}(\sigma-s)} \varphi_1(s) w_1(t) dt d\sigma , \end{array} \right.$$

that is

$$(3.34) \quad \left\{ \begin{array}{l} \varphi_1'(s) \leq C \varphi_1(s) \left(-\frac{2\sqrt{\lambda_1}}{3C^2} + \int_0^1 |g_2(s, t) + g_4(s, t)| w_1(t) dt \right. \\ \left. + \int_s^\infty \int_0^1 (|g_1(\sigma, t) + g_3(\sigma, t) + g_5(\sigma, t)| + \left| \frac{\partial(g_2+g_4)}{\partial\sigma}(\sigma, t) \right|) \right. \\ \left. e^{-\frac{\sqrt{\lambda_1}}{2}(\sigma-s)} w_1(t) dt d\sigma \right) . \end{array} \right.$$

But arguments similar to the ones in the proof of Lemma 3.6 below show that

$$(3.35) \quad \left\{ \begin{array}{l} \lim_{s \rightarrow \infty} \left\{ \int_0^1 |g_2(s, t) + g_4(s, t)| w_1(t) dt \right. \\ \left. + \int_s^\infty \int_0^1 (|g_1(\sigma, t) + g_3(\sigma, t) + g_5(\sigma, t)| + \left| \frac{\partial(g_2+g_4)}{\partial\sigma}(\sigma, t) \right|) \right. \\ \left. e^{-\frac{\sqrt{\lambda_1}}{2}(\sigma-s)} w_1(t) dt d\sigma \right\} = 0 . \end{array} \right.$$

This completes the proof. \square

Next multiply (3.28) by $\varphi_1'(s)$ and integrate from s to ∞ , obtaining

$$(3.36) \quad \left\{ \begin{array}{l} -\frac{1}{2} \varphi_1'(s)^2 + \frac{1}{2} \lambda_1 \varphi_1(s)^2 + \int_s^\infty \int_0^1 (g_1(\sigma, t) + g_3(\sigma, t) + g_5(\sigma, t)) v(\sigma, t) \varphi_1'(\sigma) d\sigma dt \\ + \int_s^\infty \int_0^1 (g_2(\sigma, t) + g_4(\sigma, t)) \frac{\partial v}{\partial\sigma}(\sigma, t) \varphi_1'(\sigma) dt d\sigma = 0 . \end{array} \right.$$

Define

$$(3.37) \quad K(s) = \int_s^\infty \int_0^1 (g_1(\sigma, t) + g_3(\sigma, t) + g_5(\sigma, t))v(\sigma, t)\varphi_1'(\sigma)dt d\sigma$$

$$(3.38) \quad L(s) = \int_s^\infty \int_0^1 (g_2(\sigma, t) + g_4(\sigma, t))\frac{\partial v}{\partial \sigma}(\sigma, t)\varphi_1'(\sigma)dt d\sigma .$$

Thus, from (3.36), (3.37) and (3.38) we get

$$(3.39) \quad \frac{\varphi_1'(s)}{\varphi_1(s)} = -\sqrt{\lambda_1 + 2\frac{K(s)}{\varphi_1^2(s)} + 2\frac{L(s)}{\varphi_1^2(s)}}$$

from which it is clear that we need estimates on $\frac{K(s)}{\varphi_1(s)^2}$ and $\frac{L(s)}{\varphi_1(s)^2}$. In particular, we shall show first that

Lemma 3.6. *Under assumption (2.4)*

$$\lim_{s \rightarrow \infty} \frac{K(s)}{\varphi_1^2(s)} = \lim_{s \rightarrow \infty} \frac{L(s)}{\varphi_1^2(s)} = 0 .$$

Proof. We first check that under assumption (2.4)

$$(3.40) \quad \lim_{s \rightarrow \infty} \sup_{0 \leq t \leq 1} |g_i(s, t)| = 0 \quad i = 1, 3, 5 .$$

For $i = 1$,

$$g_1(s, t) = \frac{d^2}{dt^2}(\tilde{c}_1(s, t)w_1(t)t^{d-2})$$

and since $w_1(t)t^{d-2}$ is bounded along with its first and second derivatives, it suffices to check that

$$(3.41) \quad \lim_{s \rightarrow \infty} \sup_{0 \leq t \leq 1} \left| \frac{d^k}{dt^k} \tilde{c}_1(s, t) \right| = 0 , \quad k = 0, 1, 2 .$$

For $k = 0$ this follows easily. For $k = 1$

$$\frac{d}{dt} \tilde{c}_1(s, t) = \frac{2tf'^2 + 2t^2f'f''x_t - \ell_t}{1 + \ell} - \frac{(t^2f'^2 - \ell)\ell_t}{(1 + \ell)^2}$$

with

$$\begin{aligned}\ell_t &= 2tk + t^2k_t - 4t^3kf'^2 - t^4k_t f'^2 - 2t^4kf'f''x_t \\ &\quad + t^3f^2f''^2 + \frac{1}{2}t^4ff'f''^2x_t + \frac{1}{2}t^4f^2f''f'''x_t \\ k_t &= (ff''' - f'f'')x_t.\end{aligned}$$

One sees using (2.4) and (3.12) that $\lim_{s \rightarrow \infty} \sup_{0 \leq t \leq 1} k_t(s, t) = 0$, so by inspection $\lim_{s \rightarrow \infty} \sup_{0 \leq t \leq 1} \ell_t = 0$ as well. Thus (2.4) implies (3.41) with $k = 1$.

For (3.41) with $k = 2$, we write down the rather unattractive expression

$$\begin{aligned}\frac{d^2\tilde{c}_1}{dt^2}(s, t) &= \frac{2f'^2 + 4tf'f''x_t + 4tf'f''x_t + 2t^2f''^2x_t^2 + 2t^2f'f^{(3)}x_t^2 - \ell_{tt}}{(1 + \ell)} \\ &\quad - \frac{(2tf'^2 + 2t^2f'f''x_t - \ell_t)\ell_t}{(1 + \ell)} \\ &\quad - \frac{(2tf'^2 + 2t^2f'f''x_t - \ell_t)\ell_t + (t^2f'^2 - \ell)\ell_{tt}}{(1 + \ell)^2} \\ &\quad + \frac{2(t^2f'^2 - \ell)\ell_t^2}{(1 + \ell)^3}\end{aligned}$$

with

$$\begin{aligned}\ell_{tt} &= 2k + 4tk_t + t^2k_{tt} - (4t^3kf'^2)_t - 4t^3k_t f'^2 - t^4k_{tt}f'^2 - 2t^4k_t f'f''x_t \\ &\quad - (2t^4kf'f''x_t)_t + (t^3f^2f''^2)_t + \frac{1}{2}(t^4ff'f''^2x_t)_t \\ &\quad + \frac{1}{2}(f^2f''f^{(3)}x_t)_t\end{aligned}$$

and

$$k_{tt} = (ff^{(4)} - f''^2)x_t^2 + (ff^{(3)} - f'f'')x_{tt}.$$

Now a simple check using the above, (3.12), (3.13), (3.17) and (3.18), reveals that (2.4) implies (3.41) for $k = 2$.

For $i = 3$,

$$g_3(s, t) = -(w_1(t)t^{d-2})'\tilde{c}_s(s, t) - w_1(t)t^{d-2}\frac{d\tilde{c}_3}{dt}(s, t),$$

so we only need check

$$(3.42) \quad \lim_{s \rightarrow \infty} \sup_{0 \leq t \leq 1} \tilde{c}_3(s, t) = 0,$$

and

$$(3.43) \quad \lim_{s \rightarrow \infty} \sup_{0 \leq t \leq 1} \left| \frac{d\tilde{c}_3}{dt}(s, t) \right| = 0.$$

That (3.42) holds under (2.4) is easily verified. As for (3.43), the only possible difficulty arises in $\frac{(d-2)\ell}{t^2}$, but $\lim_{s \rightarrow \infty} \frac{|\ell(s, t)|}{t^2} = 0$. Thus under assumption (2.4), (3.40) holds for $i = 3$.

That (3.40) holds for $i = 5$ is immediately checked. Thus, given $\epsilon > 0$, there is an $s(\epsilon)$ such that $\sup_{s \geq s(\epsilon), 0 \leq t \leq 1} |g_i(s, t)| < \epsilon$ for $i = 1, 3, 5$. Also Lemma 3.5 and (3.30) we have,

$$\int_s^\infty \varphi_1'(\sigma) \varphi_1(\sigma) d\sigma = -\frac{1}{2} \varphi_1^2(s).$$

Therefore, for $s \geq s(\epsilon)$,

$$\begin{aligned} \frac{|K(s)|}{\varphi_1^2(s)} &\leq -C3\epsilon \frac{\int_s^\infty \varphi_1(\sigma) \varphi_1'(\sigma) d\sigma}{\varphi_1^2(s)} \\ &\leq C\epsilon \end{aligned}$$

and $\lim_{s \rightarrow \infty} \frac{K(s)}{\varphi_1^2(s)} = 0$ follows.

As for $\frac{L(s)}{\varphi_1^2(s)}$, the integration by parts gives that

$$\begin{aligned}
L(s) &= \int_s^\infty \int_0^1 (g_2(\sigma, t) + g_4(\sigma, t)) \frac{\partial v}{\partial \sigma}(\sigma, t) \varphi_1'(\sigma) dt d\sigma \\
&= - \int_0^1 (g_2(s, t) + g_4(s, t)) v(s, t) \varphi_1'(s) dt \\
&\quad - \int_s^\infty \int_0^1 \left(\frac{\partial g_2}{\partial \sigma}(\sigma, t) + \frac{\partial g_4}{\partial \sigma}(\sigma, t) \right) v(\sigma, t) \varphi_1'(\sigma) dt d\sigma \\
&\quad - \int_s^\infty \int_0^1 [(g_2(\sigma, t) + g_4(\sigma, t)) v(\sigma, t) (\lambda_1 \varphi_1(\sigma) \\
&\quad - \int_0^1 (g_1(\sigma, \tau) + g_3(\sigma, \tau) + g_5(\sigma, \tau)) v(\sigma, \tau) d\tau \\
&\quad - \int_0^1 (g_2(\sigma, \tau) + g_4(\sigma, \tau)) \frac{\partial v}{\partial \sigma}(\sigma, \tau) d\tau)] d\sigma dt, \text{ by (3.30)} \\
(3.44) \quad &= - \int_0^1 (g_2(s, t) + g_4(s, t)) v(s, t) \varphi_1'(s) dt \\
&\quad - \int_s^\infty \int_0^1 \left(\frac{\partial g_2}{\partial \sigma}(\sigma, t) + \frac{\partial g_4}{\partial \sigma}(\sigma, t) \right) v(\sigma, t) \varphi_1'(\sigma) dt d\sigma \\
&\quad - \int_s^\infty \int_0^1 (g_2(\sigma, t) + g_4(\sigma, t)) v(\sigma, t) (\lambda_1 \varphi_1(\sigma) \\
&\quad \quad - \int_0^1 (g_1(\sigma, \tau) + g_3(\sigma, \tau) + g_5(\sigma, \tau)) v(\sigma, \tau) d\tau) dt d\sigma \\
&\quad - \frac{1}{2} \int_s^\infty \frac{d}{d\sigma} \left(\int_0^1 (g_2(\sigma, t) + g_4(\sigma, t)) v(\sigma, t) dt \right)^2 d\sigma \\
&\quad + \int_s^\infty \int_0^1 (g_2(\sigma, t) + g_4(\sigma, t)) v(\sigma, t) \\
&\quad \quad \left(\int_0^1 \left(\frac{\partial g_2}{\partial \sigma}(\sigma, \tau) + \frac{\partial g_4}{\partial \sigma}(\sigma, \tau) \right) v(\sigma, \tau) d\tau \right) dt d\sigma \\
&= I_1(s) + I_2(s) + I_3(s) + I_4(s) + I_5(s).
\end{aligned}$$

These terms are controlled provided we obtain appropriate bounds on $g_2(\sigma, t)$, $g_4(\sigma, t)$, $\frac{\partial g_2}{\partial \sigma}(\sigma, t)$ and $\frac{\partial g_4}{\partial \sigma}(\sigma, t)$.

For $g_2(\sigma, t)$ and $\frac{\partial g_2}{\partial s}(\sigma, t)$, it suffices to handle $\tilde{c}_2(\sigma, t)$, $\tilde{c}_2(\sigma, t)t^{d-3}$ (possibly troublesome when $d = 2$), $\frac{\partial \tilde{c}_2}{\partial t}(\sigma, t)$, and $\frac{\partial^2 \tilde{c}_2}{\partial s \partial t}(\sigma, t)$.

A quick inspection of $\tilde{c}_2(\sigma, t)$, $\tilde{c}_2(\sigma, t)t^{d-3}$ and

$$\frac{\partial \tilde{c}_2}{\partial t}(\sigma, t) = \frac{3}{t} \tilde{c}_2(\sigma, t) + \frac{t^3(5f'^2 f'' - f f''^2 - f f' f^{(3)})x_t}{1 + \ell} + \tilde{c}_2(\sigma, t) \frac{\ell_t}{1 + \ell}$$

keeping in mind that

$$(3.45) \quad \begin{cases} \ell_t(\sigma, t) = 2tk + t^2 k_t - 4t^3 k f'^2 - t^4 k_t f'^2 - 2t^4 k f' f'' x_t \\ \quad + t^3 f^2 f''^2 + \frac{1}{2} t^4 f f' f''^2 x_t + \frac{1}{2} t^4 f^2 f'' f^{(3)} x_t \\ k_t(\sigma, t) = (f f^{(3)} - f' f'') x_t . \end{cases}$$

One sees that under assumption (2.4)

$$\lim_{s \rightarrow \infty} k_t(s, t) = 0$$

and

$$\lim_{s \rightarrow \infty} \tilde{c}_2(s, t) \ell_t(s, t) = 0 .$$

Thus, from (3.12), (3.20), (3.45) and assumption (2.4) we get

$$\lim_{s \rightarrow \infty} \sup_{0 \leq t \leq 1} |g_2(s, t)| = 0 .$$

Also,

$$(3.46) \quad \begin{aligned} \frac{\partial^2 \tilde{c}_2}{\partial t \partial s}(\sigma, t) &= \frac{3}{t} \left[\frac{t^3(5f'^2 f'' - f f''^2 - f f' f^{(3)})x_s}{1 + \ell} - \tilde{c}_2(\sigma, t) \frac{\ell_s}{1 + \ell} \right] \\ &+ \frac{t^3(9f' f''^2 f^{(3)} - 3f f'' f^{(3)} - f f' f^{(4)})x_s x_t}{1 + \ell} \\ &+ \frac{t^3(t f'^2 f'' - f f''^2 - f f' f^{(3)})x_{st}}{1 + \ell} \\ &- \frac{t^3(5f'^2 f'' - f f''^2 - f f' f^{(4)})x_t \ell_s}{(1 + \ell)^2} + \frac{\partial \tilde{c}_2}{\partial s}(\sigma, t) \frac{\ell_t}{1 + \ell} \\ &+ \tilde{c}_2(\sigma, t) \frac{\ell_{st}}{1 + \ell} - \tilde{c}_2(\sigma, t) \frac{\ell_t \ell_s}{(1 + \ell)^2} , \end{aligned}$$

with

$$\frac{\partial \tilde{c}_2}{\partial s}(\sigma, t) = \frac{t^3(5f'^2 f'' - f f''^2 - f f' f^{(3)})x_s}{1 + \ell} - \tilde{c}_2(\sigma, t) \cdot \frac{\ell_s}{1 + \ell}$$

and from (3.17) and (3.18)

$$(3.47) \quad \ell_s(\sigma, t) = t^2 k_s (1 - t^2 f'^2) - 2t^4 k f'' x_s + \frac{1}{2} t^4 (f f' f''^2 + f^2 f'' f^{(3)}) x_s$$

$$(3.48) \quad k_s(\sigma, t) = (f f^{(3)} - f' f'') x_s$$

$$(3.49) \quad \begin{aligned} \ell_{st} &= 2t k_s + t^2 k_{st} - 4t^3 k_s f'^2 - 8t^3 k f' f'' x_s - t^4 k_{st} f'^2 \\ &\quad - 2t^4 k_t f' f'' x_s - 2t^4 k_s f' f'' x_t - 2t^4 k f''^2 x_s x_t - 2t^4 k f' f^{(3)} x_s x_t \\ &\quad - 2t^4 k f' f'' x_{st} + 2t^3 (f f' f''^2 + f^2 f'' f^{(3)}) x_s \\ &\quad + \frac{1}{2} t^4 (f'^2 f''^2 + f f''^3 + 4f f' f'' f''' + f^2 f^{(3)2} + f^2 f'' f^{(4)}) x_s x_t \\ &\quad + \frac{1}{2} t^4 (f f' f''^2 + f^2 f'' f^{(3)}) x_{st} \\ k_{st} &= (f f^{(4)} - f''^2) x_s x_t + (f f^{(3)} - f' f'') x_{st} . \end{aligned}$$

Examining the eight terms in (3.48) in order we see that assumption (2.4) insures that

$$\lim_{s \rightarrow \infty} \sup_{0 \leq t \leq 1} \frac{\partial^2 \tilde{c}_2}{\partial t \partial s}(s, t) = 0 .$$

As a consequence,

$$\lim_{s \rightarrow \infty} \left\{ \sup_{0 \leq t \leq 1} |g_2(s, t)| + \sup_{0 \leq t \leq 1} \left| \frac{\partial g_2}{\partial s}(s, t) \right| \right\} = 0 .$$

The verification for $g_4(s, t)$ and $\frac{\partial g_4}{\partial s}(s, t)$ is easier. It's quick to see that

$$\lim_{s \rightarrow \infty} \sup_{0 \leq t \leq 1} |g_4(s, t)| = 0$$

follows from (2.4). Also,

$$(3.50) \quad \begin{aligned} \frac{\partial g_4}{\partial s}(s, t) &= \frac{[3(d+1)t^2 f'^2 f'' - \frac{1}{2}(d+4)t^2 [f'^2 f'' + f f''^2 + f f' f^{(3)}]] x_s}{(1+\ell)} \\ &\quad + \frac{[\frac{1}{2}t^2 [2f f' f^{(3)} + f^2 f^{(4)}]] x_s}{(1+\ell)} \\ &\quad - c_4 \frac{\ell s}{1+\ell} , \end{aligned}$$

and a routine check gives

$$\lim_{s \rightarrow \infty} \sup_{0 \leq t \leq 1} \left| \frac{\partial g_4}{\partial s}(s, t) \right| = 0 .$$

Returning now to (3.43) we find using (3.29), (3.30) and Lemma 3.5 that

$$\lim_{s \rightarrow \infty} (|I_1(s)| + |I_2(s)| + |I_3(s)| + |I_4(s)| + |I_5(s)|) \varphi_1^{-2}(s) = 0 .$$

This completes the proof. □

Lemma 3.7. *Under assumption (2.4)*

$$\frac{K(s)}{\varphi_1^2(s)} , \frac{L(s)}{\varphi_1^2(s)} \in L^1[s_0, \infty) .$$

Proof. The proof uses the integrability conditions of (2.4) and follows the lines of the proof of Lemma 3.6.

The difference being we find now expressions like

$$\int_{s_0}^{\infty} \int_s^{\infty} \int_0^1 f^2 |h| v(\sigma, t) \varphi_1'(\sigma) \varphi_1^{-2}(s) dt d\sigma ds$$

with $f^2 |h| \in L^1[s_0, \infty)$, by the integrability conditions of (2.4). We proceed by observing that (3.39) and Lemma 3.6 imply that (we might need to increase s_0)

$$-\sqrt{\frac{\lambda_1}{2}} \geq \frac{\varphi_1'(s)}{\varphi_1(s)} \geq -\sqrt{\frac{3\lambda_1}{2}} , \text{ for } s \geq s_0 .$$

This implies that for $\sigma > s_0$

$$\begin{aligned} |\varphi_1'(\sigma)| \varphi_1(\sigma) \int_{s_0}^{\sigma} \frac{ds}{\varphi_1^2(s)} &\leq \sqrt{\frac{3\lambda_1}{2}} \varphi_1^2(\sigma) \int_{s_0}^{\sigma} \frac{ds}{\varphi_1^2(s)} \\ &\leq \frac{\sqrt{3}}{2} . \end{aligned}$$

Thus,

$$\begin{aligned}
& \int_{s_0}^{\infty} \int_s^{\infty} \int_0^1 f^2 |h| v(\sigma, t) |\varphi_1'(\sigma)| \varphi_1^{-2}(s) dt d\sigma ds \\
& \leq c \int_{s_0}^{\infty} f^2 \sup_{0 \leq t \leq 1} |h| \varphi_1(\sigma) |\varphi_1'(\sigma)| \int_{s_0}^{\sigma} \frac{ds}{\varphi_1^2(s)} d\sigma \quad \text{by Proposition 3.3} \\
& \leq c \sqrt{\frac{3\lambda_1}{2}} \int_{s_0}^{\infty} f^2 \sup_{0 \leq t \leq 1} |h| \varphi_1^2(\sigma) \int_{s_0}^{\sigma} \frac{ds}{\varphi_1^2(s)} d\sigma \\
& \leq \frac{c\sqrt{3}}{2} \int_{s_0}^{\infty} f^2 \sup_{0 \leq t \leq 1} |h| d\sigma < \infty .
\end{aligned}$$

□

Returning now to (3.39), we can write it in the form

$$\frac{\varphi_1'}{\varphi_1}(s) = -\sqrt{\lambda_1 + G(s)}$$

with $\lim_{s \rightarrow \infty} G(s) = 0$. Thus,

$$\frac{\varphi_1'(s)}{\varphi_1(s)} = -\sqrt{\lambda_1} + \frac{1}{\sqrt{\xi}} G(s)$$

where $\xi = \xi(s)$ is between λ_1 and $\lambda_1 + G(s)$. Then, using Lemma 3.7, $G \in L^1[s_0, \infty)$ so

$$(3.51) \quad \varphi_1(s) = \varphi_1(s_0) \exp\left\{-\sqrt{\lambda_1}(s - s_0) + \int_{s_0}^s \frac{1}{\sqrt{\xi(\sigma)}} G(\sigma) d\sigma\right\} .$$

Since $\left|s - \int_0^x \frac{d\tau}{f(\tau)}\right| = \frac{1}{2}t^2 |f'(x)| \rightarrow 0$ as $s \rightarrow \infty$, it follows immediately from (3.51) that

Proposition 3.8. *There exists a positive constant $C = C(s_0)$ such that*

$$C^{-1} \varphi_1(s_0) e^{-\sqrt{\lambda_1} \int_{x_0}^x \frac{d\tau}{f(\tau)}} \leq \varphi_1(s) \leq C \varphi_1(s_0) e^{-\sqrt{\lambda_1} \int_{x_0}^x \frac{d\tau}{f(\tau)}} .$$

We can now provide

Proof. (of Theorem 2.3) Follows from (3.10), (3.51) and Lemma 3.7. □

4 Applications

We now turn to applications of our results. First, when we take $h \equiv \mu_1$, the first Dirichlet eigenvalue of Δ_d on D and $\alpha = \beta \equiv 0$, then φ_1 , the first Dirichlet eigenfunction for Δ_d on D , satisfies (2.1). When we take $h = \alpha = \beta \equiv 0$, then solutions to (2.1) are positive harmonic functions in D vanishing on $\partial D \cap \{(x, y) : x > 0\}$. Thus we have the following (recall $\tilde{\lambda}_1$ is first Dirichlet eigenvalue of Δ_{d-1} on Ω , $z = z(x, y)$ as in Section 2.) Define for $t > 0$, $D_t = \{(x, y) \in D : x > t\}$.

Corollary 4.1. *Let Ω be as described in Section 2.*

(1) *If u solves (2.1) with $\alpha = \beta = h \equiv 0$, and (2.2) holds with this choice of α, β, h then*

$\forall \epsilon > 0$, there are constants C_1, x_1 , such that

$$C_1^{-1} f(x)^{\frac{2-d}{2}} \exp \left\{ -\sqrt{\tilde{\lambda}_1 + \epsilon} \int_0^x \frac{d\tau}{f(t)} \right\} \tilde{w}_1(z) \leq u(x, y) \leq C_1 f(x)^{\frac{2-d}{2}} \exp \left\{ -\sqrt{\tilde{\lambda}_1 - \epsilon} \int_0^x \frac{d\tau}{f(\tau)} \right\} \tilde{w}_1(z),$$

$(x, y) \in D_{x_1}$.

(2) *If φ_1 is the first Dirichlet eigenfunction for Δ_d on D , and (2.2) holds for $\alpha = \beta \equiv 0$,*

$h \equiv \mu_1$, then $\forall \epsilon > 0$, there are constants C_2, x_2 such that

$$C_2^{-1} f(x)^{\frac{2-d}{2}} \exp \left\{ -\sqrt{\tilde{\lambda}_1 + \epsilon} \int_0^x \frac{d\tau}{f(\tau)} \right\} \tilde{w}_1(z) \leq \varphi_1(x, y) \leq C_2 f(x)^{\frac{2-d}{2}} \exp \left\{ -\sqrt{\tilde{\lambda}_1 - \epsilon} \int_0^x \frac{d\tau}{f(\tau)} \right\} \tilde{w}_1(z),$$

for $(x, y) \in D_{x_2}$.

When D is the horn-shaped region based on $\Omega = B(0, 1) \subset \mathbb{R}^{d-1}$ and Theorem 2.3 holds, take λ_1 to be the first Dirichlet eigenvalue for Δ_{d-1} on $B(0, 1)$ and w_1 the corresponding eigenfunction, then

Corollary 4.2. *If $D = \{(x, y) : x > 0, \|y\| < f(x)\}$ and f satisfies (2.4), then*

(1) *If $\Delta_d u = 0$, $u > 0$ on D , $u = 0$ on $\partial D \cap \{(x, y) : x > 0\}$ and $\lim_{x \rightarrow \infty} u(x, 0) = 0$, then*

there are constants C_1, x_1 such that

$$\begin{aligned} & C_1^{-1} f(x)^{\frac{2-d}{2}} \exp \left\{ -\sqrt{\lambda_1} \int_0^x \frac{d\tau}{f(\tau)} \right\} w_1 \left(\frac{\|y\|}{f(x)} \right) \leq u(x, y) \\ & \leq C_1 f(x)^{\frac{2-d}{2}} \exp \left\{ -\sqrt{\lambda_1} \int_0^x \frac{d\tau}{f(\tau)} \right\} w_1 \left(\frac{\|y\|}{f(x)} \right), \end{aligned}$$

for $(x, y) \in D_{x_1}$.

(2) *If φ_1 is the first Dirichlet eigenfunction of Δ_d on D there are constants C_2 and x_2 such*

that

$$\begin{aligned} & C_2^{-1} f(x)^{\frac{2-d}{2}} \exp \left\{ -\sqrt{\lambda_1} \int_0^x \frac{d\tau}{f(\tau)} \right\} w_1 \left(\frac{\|y\|}{f(x)} \right) \leq \varphi_1(x, y) \\ & \leq C_2 f(x)^{\frac{2-d}{2}} \exp \left\{ -\sqrt{\lambda_1} \int_0^x \frac{d\tau}{f(\tau)} \right\} w_1 \left(\frac{\|y\|}{f(x)} \right), \end{aligned}$$

for $(x, y) \in D_{x_2}$.

These estimates also give information when the operator $L = \Delta_d + \alpha \frac{\partial}{\partial x} + \beta \nabla_y + h$ corresponds to an intrinsically ultracontractive (IU) semigroup. The reader is referred to the article of Davies and Simon [5] on this subject. The semigroup e^{-tL} is (IU) if $\varphi_1^{-1} e^{-t(L-\mu_1)} \varphi_1$ maps $L^2(D, \varphi_1^2(x) dx)$ into $L^\infty(D, \varphi_1^2(x) dx)$ for all $t > 0$.

As outlined in [5] whether e^{-tL} is (IU) depends on the behavior of φ_1 . A sufficient condition for e^{-tL} to be (IU) in our context (see sections 7, 8 and 9 of [5]) is

$$(4.1) \quad -\log \varphi_1 \leq \epsilon f^{-2} + e^{\epsilon^{-a}}, \text{ for some } a > 0, \text{ for all small } \epsilon.$$

In the case of L satisfying the conditions of (2.2), by Theorem 2.1, (4.1) holds if

$$(4.2) \quad \sqrt{\tilde{\lambda}_1 + \delta} \int_0^x \frac{d\tau}{f(\tau)} \leq \epsilon f(x)^{-2} + e^{\epsilon^{-a}}, \text{ for some } a > 0, \text{ for all small } \epsilon.$$

So, for example, (4.2) holds if $f(x) = (x + 1)^{-s}(\ln(x + 2))^{-t}$ when $s = 1$ and $t > 1$ or $s > 1$, as was pointed out in [5]. A few of the consequences of Theorem 2.1 and (4.2) are outlined in the following corollary. These are consequences of e^{-tL} being IU and the reader is referred to [5] for details and for numerous other consequences.

Corollary 4.3. *Assume (2.3) with $\alpha = \beta \equiv 0$ and (4.2) hold. Let φ_n be the n^{th} Dirichlet eigenfunction and $p(t, \cdot, \cdot)$ the heat kernel for $\Delta_d + h$ on D . Then there are $C_0(t), C_n, n \geq 2$ such that*

- (1) $C_0^{-1}(t) \exp \left\{ -\sqrt{\lambda_1 + \delta} \left[\int_0^x \frac{d\tau}{f(\tau)} + \int_0^s \frac{d\tau}{f(\tau)} \right] \right\} w_1(z)w_1(u) \leq p(t, (x, z), (s, u)) \leq C_0(t) \exp \left\{ -\sqrt{\lambda_1 - \delta} \left[\int_0^x \frac{d\tau}{f(\tau)} + \int_0^s \frac{d\tau}{f(\tau)} \right] \right\} w_1(z)w_1(u),$ for $(x, z), (s, u) \in D$.
- (2) $|\varphi_n(x, z)| \leq C_n \exp \left\{ -\sqrt{\lambda_1 - \delta} \int_0^x \frac{d\tau}{f(\tau)} \right\} w_1(z).$

In the case when, $\Omega = B(0, 1)$ we replace (4.2) by

$$(4.3) \quad \sqrt{\lambda_1} \int_0^x \frac{d\tau}{f(\tau)} \leq \epsilon f(x)^{-2} + e^{\epsilon^{-a}} \text{ for some } a > 0, \text{ all small } \epsilon > 0.$$

Then we obtain

Corollary 4.4. *Assume (2.4) with $\alpha = \beta \equiv 0$ and (4.3) hold. Let φ_n be the n^{th} Dirichlet eigenfunction and $p(t, \cdot, \cdot)$ the heat kernel for $\Delta_d + h$ on D . Then there are constants $C_0(t), C_n, n \geq 2$ such that*

- (1) $C_0^{-1}(t) f(x)^{\frac{2-d}{2}} f(s)^{\frac{2-d}{2}} \exp \left\{ -\sqrt{\lambda_1} \left[\int_0^x \frac{d\tau}{f(\tau)} + \int_0^s \frac{d\tau}{f(\tau)} \right] \right\} w_1(z)w_1(u) \leq p(t, (x, z), (s, u)) \leq C_0(t) f(x)^{\frac{2-d}{2}} f(s)^{\frac{2-d}{2}} \exp \left\{ -\sqrt{\lambda_1} \left[\int_0^x \frac{d\tau}{f(\tau)} + \int_0^s \frac{d\tau}{f(\tau)} \right] \right\} w_1(z)w_1(u),$ for $(x, z), (s, u) \in D$.
- (2) $|\varphi_n(x, z)| \leq C_n f(x)^{\frac{2-d}{2}} \exp \left\{ -\sqrt{\lambda_1} \int_0^x \frac{d\tau}{f(\tau)} \right\} w_1(z), (x, z) \in D.$

Finally, we apply our results to bounded domains with cusps. Now let $f : [0, \infty) \rightarrow (0, \infty)$ and define $D = \{(x, y) : x^2 + \|y\|^2 > 1, x > 0, \|y\| < f(x)\}$. Using the Kelvin transformation we can transfer the results of Section 2 to the present setting. For $\xi \in \mathbb{R}^d$, $\xi = (x, y)$, put $\xi' = \frac{\xi}{\|\xi\|^2}$, $\xi' = (x', y')$, this is inversion in S^{d-1} . Given D , $D' = \{\xi' : \xi \in D\}$ has a cusp at the origin we can now see how harmonic functions on D' which vanish in a neighborhood of the origin on $\partial D'$ and the first eigenfunction decay in the cusp. Let $u' \in C^2(D')$, then $u \in C^2(D)$ where $u'(\xi') = u(\xi)$ and

$$\Delta'_d u'(\xi') = \|\xi\|^{d+2} \Delta_d \left(\frac{u(\xi)}{\|\xi\|^{d-2}} \right)$$

where Δ'_d denotes the Laplacian in the ξ' -variable. Let μ_1 be the first Dirichlet eigenvalue for D . Thus $\Delta'_d u'_1 + \mu_1 u'_1 = 0$ in D' if and only if $\Delta_d \left(\frac{u_1(\xi)}{\|\xi\|^{d-2}} \right) + (\|\xi\|^{-4} \mu_1) \frac{u_1(\xi)}{\|\xi\|^{d-2}} = 0$ in D and, of course, u' is harmonic on D' if and only if $\frac{u(\xi)}{\|\xi\|^{d-2}}$ is harmonic on D . Transferring Corollary 4.2 to the present situation yields

Theorem 4.5. *Suppose $f(x)$ satisfies (2.4) with $h = (x+1)^{-4}$ and let $D = \{(x, y) : x^2 + \|y\|^2 > 1, x > 0, \|y\| < f(x)\}$. The image of D under the Kelvin transform with respect to the unit ball in \mathbb{R}^d will be denoted D' . Then $D' = \{(x', y') : (x')^2 + \|y'\|^2 < 1, x' > 0, \frac{\|y'\|}{(x')^2 + \|y'\|^2} < f\left(\frac{x'}{(x')^2 + \|y'\|^2}\right)\}$.*

(1) *If u' is harmonic on D ; $u' > 0$, $u' = 0$ on $\partial D' \cap \{(x', y') : (x')^2 + \|y'\|^2 < 1\}$, then there*

exists C such that with $\xi' = (x', y')$,

$$\begin{aligned} \text{(a)} \quad & C^{-1} (\|\xi'\|^2 f\left(\frac{x'}{\|\xi'\|^2}\right))^{\frac{2-d}{2}} \exp\left\{-\sqrt{\lambda_1} \int_0^{\frac{x'}{\|\xi'\|^2}} \frac{d\tau}{f(\tau)}\right\} w_1\left(\frac{\|y'\|}{\|\xi'\|^2 \cdot f\left(\frac{x'}{\|\xi'\|^2}\right)}\right) \leq u'(x', y') \leq \\ & C \left(\|\xi'\|^2 f\left(\frac{x'}{\|\xi'\|^2}\right)\right)^{\frac{2-d}{2}} \exp\left\{-\sqrt{\lambda_1} \int_0^{\frac{x'}{\|\xi'\|^2}} \frac{d\tau}{f(\tau)}\right\} w_1\left(\frac{\|y'\|}{\|\xi'\|^2 \cdot f\left(\frac{x'}{\|\xi'\|^2}\right)}\right). \end{aligned}$$

In particular, when $y' = 0$,

$$\begin{aligned} \text{(b)} \quad & C^{-1} (x')^2 f\left(\frac{1}{x'}\right)^{\frac{2-d}{2}} \exp\left\{-\sqrt{\lambda_1} \int_0^{1/x'} \frac{d\tau}{f(\tau)}\right\} w_1(0) \leq u'(x', 0) \leq C (x')^2 f\left(\frac{1}{x'}\right)^{\frac{2-d}{2}} \\ & \exp\left\{-\sqrt{\lambda_1} \int_0^{1/x'} \frac{d\tau}{f(\tau)}\right\} w_1(0). \end{aligned}$$

(2)] If φ'_1 is the first Dirichlet eigenfunction on D' with corresponding eigenvalue μ_1 , then
1(a) and 1(b) hold with φ'_1 in place of u' .

So, for example, Theorem 4.5 holds if $f(x') = x'^{(2+\alpha)}(-\ln(x'))^{-\beta}$ when $\alpha = 1$ and $\beta > 1$ or $\alpha > 1$, with domains $D' = \{(x', y') : x'^2 + \|y'\|^2 < 1, x' > 0, \|y'\| < f(x')\}$.

5 Appendix

In this section we shall give some general comparison property and monotone property of positive solutions of (2.1) and (2.3). Let D, \tilde{w}_1 and $\tilde{\lambda}_1$ be as in Section 2. We will use \tilde{D}_x for a fixed $x > 0$ to denote the cross section of D at x , i.e. the set of y such that $(x, y) \in D$.

Lemma 5.1. *If u solves (2.3) and $u(0, y) = u(0, \|y\|)$ is nonincreasing in $\|y\|$ with $h(x, y) = h(x, \|y\|)$ nonincreasing in $\|y\|$, then*

$$u(x, y) = u(x, \|y\|) \text{ and } u(x, \|y\|) \text{ is decreasing in } \|y\| \text{ for } (x, y) \in D .$$

Proof. This result can be proved by the standard moving plane argument used in [6]. Thus we omit its proof here. □

Lemma 5.2. *If u is a subsolution of (2.1) and v is a supersolution of (2.1) where α, β, h and R, S satisfy (2.2), then there exists $x_0 > 0$ such that for $C = \sup_{D_{x_0}} \frac{u(x_0, y)}{v(x_0, y)}$*

$$u(x, y) \leq Cv(x, y) \text{ for } (x, y) \in D \cap \{(x, y) : x > x_0\} .$$

Proof. Since v is a supersolution of (2.1) and $v = 0$ on $\partial D \cap \{(x, y) : x > 0\}$ the outer normal derivative of v on $\partial D \cap \{(x, y) : x > 0\}$ is negative everywhere. Hence C is well-defined for all $x_0 > 0$. For any $M > 0$, $u - Mv$ is a subsolution of (2.1) in D . Then

$$w\left(\int_0^x \frac{d\tau}{f(\tau)}, \frac{R(x)(y - S(x))}{f(x)}\right) = (u - Mv)(x, y) .$$

satisfies for $(s, z) \in (0, \infty) \times \Omega$ with $s = \int_0^x \frac{d\tau}{f(\tau)}$ and $z = \frac{R(x)(y-S(x))}{f(x)}$ that if D_\cdot denotes the derivatiave with respect to \cdot , then

$$(5.1) \quad \begin{aligned} \mathcal{A}w = & \Delta_d w + D_z^2 w(fD_x z, fD_x z) + \langle \nabla_{sz}^2 w, fD_x z \rangle + (\alpha f - f')D_s w \\ & + \langle \nabla_z w, f^2 D_x^2 z \rangle + \alpha f \langle \nabla_z w, fD_x z \rangle + \langle f\beta, \nabla_z w fD_y z \rangle + f^2 h w \leq 0 . \end{aligned}$$

Fix R large enough so that $\bar{\Omega} \in B(0, R) \subset \mathbb{R}^{d-1}$ and let $w_R(y)$ and $\lambda_1(R)$ be the first Dirichlet eigenfunction and eigenvalue for Δ_d over $B(0, R)$. Then $\frac{w}{w_R}$ satisfies

$$(5.2) \quad \left\{ \begin{aligned} & \Delta_d \left(\frac{w}{w_R} \right) + D_z^2 \left(\frac{w}{w_R} \right) (fD_x z, fD_x z) + \langle \nabla_{sz}^2 \left(\frac{w}{w_R} \right), fD_x z \rangle + (\alpha f - f') D_s \left(\frac{w}{w_R} \right) \\ & + \langle \nabla_z \left(\frac{w}{w_R} \right), f^2 D_x^2 z \rangle + \alpha f \langle \nabla_z \left(\frac{w}{w_R} \right), fD_x z \rangle + \langle f\beta, \nabla_z \left(\frac{w}{w_R} \right) fD_y z \rangle \\ & + \frac{\langle D \left(\frac{w}{w_R} \right), Dw_R \rangle + \langle D_z \left(\frac{w}{w_R} \right), D_z w_R \rangle (fD_x z, fD_x z)}{w_R} \\ & + (-\lambda_1(R) + f^2 h + \frac{D_z^2 w_R (fD_x z, fD_x z) + \langle \nabla_z w_R, f^2 D_x^2 z \rangle + \alpha f \langle \nabla_z w_R, fD_x z \rangle}{w_R} \\ & + \frac{\langle f\beta, \nabla_z w_R fD_y z \rangle}{w_R}) \left(\frac{w}{w_R} \right) \leq 0 , \\ & \frac{w}{w_R} = 0 \text{ on } (0, \infty) \times \partial\Omega . \end{aligned} \right.$$

From (2.2) it is clear that there exists $s_0 > 0$ such that in $(s_0, \infty) \times \Omega$

$$f^2 h + \frac{D_z^2 w_R (fD_x z, fD_x z) + \langle \nabla_z w_R, f^2 D_x^2 z \rangle + \alpha f \langle \nabla_z w_R, fD_x z \rangle + \langle f\beta, \nabla_z w_R fD_y z \rangle}{w_R} \leq \lambda_1(R) .$$

Therefore the maximum principle applies here to yield that

$$\frac{w}{w_R} \leq \max \left\{ \sup_{\Omega} \left\{ \frac{w(s_0, z)}{w_R(s_0, z)} \right\}, 0 \right\} \text{ for } (s, z) \in (s_0, \infty) \times \Omega .$$

Finally taking $x_0 > 0$ with $s_0 = \int_0^{x_0} \frac{d\tau}{f(\tau)}$ and $M = C$ completes the proof. \square

Lemma 5.3. *Let α, β, h and R, S satisfy (2.2) in D . Then there exists some $x_1 \geq x_0$ such that if $\psi(y)$ is a positive function in $C^1(\tilde{D}_{x_1})$ and $\psi = 0$ on $\partial\tilde{D}_{x_1}$ with negative outer normal*

derivative on $\partial\tilde{D}_{x_1}$, then the following problem

$$(5.3) \quad \left\{ \begin{array}{l} Lu = 0, \text{ on } D \cap \{(x, y) : x > x_1\}, \\ u > 0, \text{ on } D \cap \{(x, y) : x > x_1\}, \\ u = 0, \text{ on } \partial D \cap \{(x, y) : x > x_1\}, u(x_1, y) = \psi(y), \text{ on } \tilde{D}_{x_1}, \\ \lim_{x \rightarrow \infty} u(x, y) = 0. \end{array} \right.$$

has a unique solution u . Furthermore there exists some constant $C = C(x_1)$ such that for $x > x_1$ we have

$$(5.4) \quad C^{-1} \inf_{\tilde{D}_{x_1}} \left\{ \frac{\psi}{\tilde{w}_1} \right\} e^{-\frac{3\sqrt{\tilde{\lambda}_1}}{2} \int_{x_1}^x \frac{d\tau}{f(\tau)}} \tilde{w}_1(z) \leq u(x, y) \leq C \sup_{\tilde{D}_{x_1}} \left\{ \frac{\psi}{\tilde{w}_1} \right\} e^{-\frac{\sqrt{\tilde{\lambda}_1}}{2} \int_{x_1}^x \frac{d\tau}{f(\tau)}} \tilde{w}_1(z).$$

Proof. Fix some $\epsilon < \epsilon_0$ given in Lemma 3.2 so that Proposition 3.1 implies

$$u_-(x, y) = \exp \left\{ -\frac{3\sqrt{\tilde{\lambda}_1}}{2} \int_{x_1}^x \frac{d\tau}{f(\tau)} \right\} \tilde{w}_1^{-\epsilon}(z)$$

and

$$u_+(x, y) = \exp \left\{ -\frac{\sqrt{\tilde{\lambda}_1}}{2} \int_{x_1}^x \frac{d\tau}{f(\tau)} \right\} \tilde{w}_1^{+\epsilon}(z)$$

are sub- and super-solutions of (5.3), respectively, for L on $D \cap \{(x, y) : x > x_1\}$ for some $x_1 \geq x_0$.

From the assumptions on ψ we can define

$$(5.5) \quad a = \inf_{\tilde{D}_{x_1}} \left\{ \frac{\psi}{\tilde{w}_1^{-\epsilon}} \right\} \text{ and } b = \sup_{\tilde{D}_{x_1}} \left\{ \frac{\psi}{\tilde{w}_1^{+\epsilon}} \right\}.$$

Then the monotone iteration method (see Theorem 2.10 in [8]) yields a solution u of (5.3)

satisfying

$$(5.6) \quad a \exp \left\{ -\frac{3\sqrt{\tilde{\lambda}_1}}{2} \int_{x_1}^x \frac{d\tau}{f(\tau)} \right\} \tilde{w}_1^{-\epsilon}(z) \leq u \leq b \exp \left\{ -\frac{\sqrt{\tilde{\lambda}_1}}{2} \int_{x_1}^x \frac{d\tau}{f(\tau)} \right\} \tilde{w}_1^{+\epsilon}(z).$$

Since $x_1 \geq x_0$ Lemma 5.2 implies that such solution is uniqueness. Finally Lemma 3.2, (5.5) and (5.6) implies (5.4), which completes the proof. □

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