EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS FOR A SEMILINEAR ELLIPTIC EQUATION

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ABSTRACT. In this paper, we consider the semilinear elliptic problem

$$-\triangle u + u = |u|^{p-2}u + \mu f(x), \quad u \in H^1(\mathbb{R}^N), \quad N > 2. \qquad ((*)_\mu)$$

For $p > 2$, we show that there exists a positive constant $\mu^* > 0$ such that $(*)_\mu$ possesses a minimal positive solution if $\mu \in (0, \mu^*)$ and no positive solutions if $\mu > \mu^*$. Furthermore, if $p < \frac{2N}{N-2}$, then $(*)_\mu$ possesses at least two positive solutions for $\mu \in (0, \mu^*)$, an unique positive solution if $\mu = \mu^*$ and there exists a constant $\mu_* > 0$ such that when $\mu \in (0, \mu_*)$, problem $(*)_\mu$ possesses at least three solutions. We also obtain some bifurcation results of the solutions at $\mu = 0$ and $\mu = \mu^*$.

§1 Introduction

In this paper we first consider the semilinear elliptic problem

$$-\triangle u + u = u^{p-1} + \mu f(x), \quad x \in \mathbb{R}^N, \quad N > 2 \qquad ((1.1)_\mu)$$

$$u \in H^1(\mathbb{R}^N), \qquad u > 0 \quad in \quad \mathbb{R}^N, \qquad (1.2)$$

where $\mu \geq 0$, $p > 2$ and $f(x)$ is some given function in $H^{-1}(\mathbb{R}^N)$ such that $f(x) \geq 0$, $f(x) \not\equiv 0$ in $\mathbb{R}^N$.

Recently, many authors have studied the existence of positive solutions of the semilinear elliptic boundary value problem

$$\begin{cases} -\triangle u + u = g(x,u), & x \in \Omega \subset \mathbb{R}^N, \\ u|_{\partial\Omega} = 0. \end{cases} \qquad (1.3)$$

a problem that occurs in various branches of mathematical physics. There are many results about the existence of the positive solutions of (1.3) when $g(x,u)$ is a "homogeneous" function (i.e. $g(x,0) \equiv 0$), see [4, 5, 9, 16, 22, 23, 25]. For the "inhomogeneous" case (i.e. $g(x,0) \not\equiv 0$), some existence of two solutions have been obtained in [15] when

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$g(x,u)$ is less than critical growth in the sense that $\lim_{u\to+\infty}\frac{g(x,u)}{u^q}=0$ with $q=\frac{N+2}{N-2}$ and Ω is bounded. A substantial difference between the problems on bounded domain and on unbounded domain is the lack of compactness for Sobolev embedding when we deal with the later. Thus there seems to be little progress on the existence theory for the "inhomogeneous" case of (1.3) when Ω is unbounded. Zhu and Zhou in their recent work [29] have obtained the existence of two positive solutions of the problem

$$-\triangle u+u=\lambda(g(u)+f(x)),\quad u\in H_0^1(\Omega)$$

by using variational and barrier methods when $\Omega=\mathbb{R}^N\setminus\omega$, and $\omega\subset\mathbb{R}^N$ is a bounded non-empty smooth domain. A similar result has also been obtained in [27] for problem

$$-\triangle u+u=\lambda f(u+\phi),\quad u\in H_0^1(\Omega).$$

The growth of nonlinear function f and g are required to be lower than the critical exponent in both papers.

The principal aim of this paper is to get the existence and nonexistence of multiple solutions for problem $(1.1)_\mu$-(1.2) for $\mu\in(0,\ +\infty)$. We also get some bifurcation results of solutions at $\mu=0$ and $\mu=\mu^*$, where μ^* is given in Theorem 1 below. It should be noted that we discuss $(1.1)_\mu$-(1.2) without the growth condition on u^{p-1} when we obtain the minimal solutions. Furthermore, for the following problem

$$\begin{cases}-\triangle u+u=|u|^{p-2}u+\mu f(x)\\ u\in H^1(\mathbb{R}^N)\end{cases}\qquad ((1.1)^*_\mu)$$

we can prove that it has at least three solutions if $\mu\in(-\mu_*,\ \mu_*)$, where μ_* is a positive constant. Such kind of result has been obtained in [3] by perturbation method. In fact, it has been proved in [3] that problem $(1.1)^*_\mu$ has infinite many solutions if $2<p<p_N<\frac{2N}{N-2}$, where p_N-1 is the largest root of the equation $(2N-2)q^2-(N+2)q-N=0$. When $p\geq p_N$, to the best of our knowledge, there is no existence result about the multiple solutions for $(1.1)^*_\mu$-(1.2). We obtain the third solution for $(1.1)^*_\mu$-(1.2) by using implicit function theorem. The results of this paper are stated in the following theorems

Theorem 1. *Let $|x|^{N-2}f(x)$ be bounded. Then there exists a constant $\mu^*>0$ such that*

i) $(1.1)_\mu$-(1.2) possesses a minimal solution u_μ for all $\mu\in(0,\mu^*)$ and $p>2$ and u_μ is increasing with respect to μ;

ii) $(1.1)_\mu$-(1.2) possesses an unique solution if $\mu=\mu^*$ and $p\in(2,\frac{2N}{N-2}]$;

iii) there are no solutions of $(1.1)_\mu$-(1.2) for $\mu>\mu^*$.

Furthermore

$$\begin{aligned}\mu_1\equiv&\frac{(N(N-2))^{\frac{N-2}{2}}(p-1)^{\frac{3-p}{p-2}}(p-2)}{\sup\limits_{x\in\mathbb{R}^N}\{(N(N-2)(p-1)^{\frac{4}{(N-2)(p-2)}}+|x|^2)^{\frac{N-2}{2}}f(x)\}}\\ &\leq\mu^*<\inf_{\epsilon>0}\{\frac{C_\epsilon S^{\frac{N}{2}}}{\int_{\mathbb{R}^N}f(x)w_\epsilon^{2^*}(x)\,dx}\}\equiv\mu_2,\end{aligned}\qquad(1.4)$$

where

$$\begin{cases} w_\epsilon = (N(N-2)\epsilon)^{\frac{N-2}{4}}(\frac{1}{\epsilon+|x|^2})^{\frac{N-2}{2}}, & \epsilon>0, \\ C_\epsilon = \frac{4}{N-2}\left(\frac{(2N^2\epsilon^{-1}+1)(N-2)}{(N+2)}\right)^{\frac{N+2}{4}} \end{cases} \tag{1.5}$$

and S is the Sobolev constant for the embedding $H_0^1(\mathbb{R}^N)\hookrightarrow L^{2^*}(\mathbb{R}^N)$ and $2^*=\frac{2N}{N-2}$.

Theorem 2. *Suppose that $|x|^{N-2}f(x)$ is bounded and $p\in(2,\ \frac{2N}{N-2})$. Then $(1.1)_\mu$-(1.2) possess at least two solutions for all $\mu\in(0,\mu^*)$.*

Theorem 3. *Under the assumption of theorem 2, we can find a constant $\mu_*>0$ such that problem $(1.1)_\mu^*$ has at least three solutions if $\mu\in(0,\mu_*)$ and two of them are positive.*

Theorem 4. *Let $p\in(2,\ \frac{2N}{N-2})$ and $f(x)\in C^\alpha\cap L^2(\mathbb{R}^N)$ with $|x|^{N-2}f(x)$ being bounded in $\mathbb{R}^N$. Then*

i) The set of solutions

$$U=\{u\in H^1(\mathbb{R}^N)\ \ |\ \ u \ is\ a\ solution\ of\ (1.1)_\mu-(1.2)\} \tag{1.6}$$

is bounded uniformly in $L^\infty(\mathbb{R}^N)$.

ii) u_μ is continuous and increasing with respect to μ if $\mu\in(0,\ \mu^*)$ for all $x\in\mathbb{R}^N$.

iii) $(\mu^*,\ u_{\mu^*})$ is in $C^{2,\alpha}(\mathbb{R}^N)\cap H^2(\mathbb{R}^N)$ and is a bifurcation point for $(1.1)_\mu$-(1.2) and

$$u_\mu\longrightarrow 0\ \ in\ \ C^{2,\alpha}(\mathbb{R}^N)\cap H^2(\mathbb{R}^N)\ \ as\ \ \mu\longrightarrow 0,$$

$$U_\mu\longrightarrow U_0\ \ in\ \ C^{2,\alpha}(\mathbb{R}^N)\cap H^2(\mathbb{R}^N)\ \ as\ \ \mu\longrightarrow 0,$$

where U_0 is the unique positive solution of $(1.1)_0$-(1.2), u_μ is the minimal solution of $(1.1)_\mu$-(1.2) and U_μ is the second solution of $(1.1)_\mu$-(1.2) constructed in Theorem 2.

We prove Theorem 1 by means of a standard barrier method and Theorem 2 by variational method. Finally we obtain Theorem 3, Theorem 4 by bifurcation theory. Similar results like Theorem 1 and 2 have been obtained in [13] on bounded domains.

§2 Proof of Theorem 1

In this section, we discuss the existence of the minimal solution of $(1.1)_\mu$-(1.2) by using standard barrier methods.

Lemma 2.1. *Under the assumption of Theorem 1, The problem $(1.1)_\mu$-(1.2) possesses a minimal solution for all $\mu\in(0,\mu_1)$.*

Proof.

For $\epsilon>0$, set

$$w_\epsilon=(N(N-2)\epsilon)^{\frac{N-2}{4}}(\frac{1}{\epsilon+|x|^2})^{\frac{N-2}{2}}. \tag{2.1}$$

Then $w_\epsilon(x)$ satisfies (see [2])

$$-\triangle w_\epsilon = w_\epsilon^{2^*-1} \qquad x\in\mathbb{R}^N, \tag{2.2}$$

and

$$|\nabla w_\epsilon|_2^2 = |w_\epsilon|_{2^*}^{2^*} = S^{\frac{N}{2}}, \tag{2.3}$$

where S is the best Sobolev constant, $2^* = \frac{2N}{N-2}$ is the critical exponent and $|\cdot|_p$ denotes the $L^p - norm$ in $\mathbb{R}^N$. We choose $\epsilon = N(N-2)(p-1)^{\frac{4}{(p-2)(N-2)}}$ and let $\tilde{u} = w_\epsilon$. Then

$$-\triangle \tilde{u} + \tilde{u} - \tilde{u}^{p-1} - \mu f(x) = w_\epsilon(1 - w_\epsilon^{p-2} + w_\epsilon^{2^*-2} - \frac{\mu f(x)}{w_\epsilon}). \tag{2.4}$$

¿From the assumption of Theorem 1 we deduce that $(\epsilon + |x|^2)^{\frac{N-2}{2}} f(x)$ is bounded. So we define

$$M = \sup_{x\in\mathbb{R}^N} \{(\epsilon + |x|^2)^{\frac{N-2}{2}} f(x)\}, \quad \mu_1 = \frac{(N(N-2))^{\frac{N-2}{2}}(p-1)^{\frac{3-p}{p-2}}(p-2)}{M}. \tag{2.5}$$

We then have

$$\begin{aligned}
& w_\epsilon(1 - w_\epsilon^{p-2} + w_\epsilon^{2^*-2} - \mu\frac{f(x)}{w_\epsilon}) \\
& \geq w_\epsilon(1 - w_\epsilon^{p-2}(0) - \mu\frac{f(x)}{w_\epsilon}) \\
& = w_\epsilon(\frac{p-2}{p-1} - \mu\frac{f(x)}{w_\epsilon}) \\
& \geq (N(N-2)\epsilon)^{\frac{N-2}{2}}(\epsilon + |x|^2)^{-\frac{N-2}{2}}\left[(N(N-2)\epsilon)^{\frac{N-2}{4}}\frac{p-2}{p-1} - \mu M\right] \geq 0
\end{aligned}$$

if $\mu \leq \mu_1$. Thus $\tilde{u} = w_\epsilon$ is a supersolution of $(1.1)_\mu$ if $\mu \in (0, \mu_1]$. On the other hand, $\underset{\sim}{u} = 0$ is clearly a subsolution of $(1.1)_\mu$ for all $\mu > 0$ and $\underset{\sim}{u} < \tilde{u}$. By the standard barrier method (see [1] Theorem 9.4 or [19]) there exists a solution u_μ of $(1.1)_\mu$ such that $0 \leq u_\mu \leq \tilde{u}$. Since 0 is not a solution of $(1.1)_\mu$ and $f(x) \geq 0$, the maximum principle implies that $0 < u_\mu \leq \tilde{u}$. Again using a result of Amann (see [1], Theorem 9.4 or [19]) we can choose a minimal solution u_μ in the order interval $[0, \tilde{u}]$ by a iteration scheme with initial value $u_{(0)} = \underset{\sim}{u} \equiv 0$.

Now we show that u_μ is minimal among all solutions of $(1.1)_\mu$-(1.2). In fact, let u be any other solution of $(1.1)_\mu$-(1.2), then $\tilde{u}^* = u$ may be considered as a supersolution of $(1.1)_\mu$. Clearly, $\underset{\sim}{u}^* = 0$ is a subsolution of $(1.1)_\mu$. By using the result of Amann we can obtain a minimal solution u_μ^* in the order interval [0, u] by an iteration scheme with initial value $u_{(0)} = \underset{\sim}{u}^* = 0$. Because $\underset{\sim}{u}^* = \underset{\sim}{u} = 0$ we deduce that $u_\mu^* \equiv u_\mu$. Thus

$$0 = \underset{\sim}{u} < u_\mu \equiv u_\mu^* \leq \tilde{u}^* \equiv u.$$

Since

$$\begin{aligned}
& \int_{\mathbb{R}^N} (|\nabla u_\mu|^2 + u_\mu^2)\, dx = \int_{\mathbb{R}^N} u_\mu^p\, dx + \mu \int_{\mathbb{R}^N} f(x) u_\mu\, dx \\
& \leq \int_{\mathbb{R}^N} \tilde{u}^p\, dx + \mu \int_{\mathbb{R}^N} f(x)\tilde{u}\, dx < +\infty
\end{aligned}$$

we deduce that $u_\mu \in H^1(\mathbb{R}^N)$ □

Remark 2.1. ¿From the proof of Lemma 2.1 we conclude that

$$1-(p-1)u_\mu^{p-2}\geq 0 \quad for \ \ \mu\in(0,\mu_1]. \tag{2.6}$$

In fact, since $\epsilon = N(N-2)(p-1)^{\frac{4}{(N-2)(p-2)}}$ we have

$$w_\epsilon \leq (N(N-2))^{\frac{N-2}{4}}\epsilon^{\frac{-(N-2)}{4}} = (\frac{1}{p-1})^{\frac{1}{p-2}}.$$

If $\mu\in(0,\mu_1]$, and u_μ is the minimal positive solution of $(1.1)_\mu$, then $0<u_\mu<\widetilde{u}= w_\epsilon\leq(\frac{1}{p-1})^{\frac{1}{p-2}}$, which gives (2.6).

Remark 2.2. Actually for the existence of the minimal solution when $N>5$, the condition on f can be weakened to that of $|x|^{\frac{N+2}{2}}f(x)$ being bounded.

Lemma 2.2. *The problem* $(1.1)_\mu$*-(1.2) has no solutions if* $\mu>\mu_2$*, where* μ_2 *is given by (1.4).*

Proof.

Let u be a positive solution of $(1.1)_\mu$-(1.2). Then for any $\epsilon>0$

$$-\triangle u w_\epsilon^{2^*}+uw_\epsilon^{2^*}=u^{p-1}w_\epsilon^{2^*}+\mu f(x)w_\epsilon^{2^*}. \tag{2.7}$$

Since $p>2$ we may conclude that for any $M>0$ there exists a constant $C>0$ such that

$$u^{p-1}\geq Mu-C \quad for \ \ all \ \ u>0. \tag{2.8}$$

It follows from (2.7), (2.8) that

$$-\int_{\mathbb{R}^N}\triangle u w_\epsilon^{2^*}dx+\int_{\mathbb{R}^N}uw_\epsilon^{2^*}dx\geq\int_{\mathbb{R}^N}((Mu-C)w_\epsilon^{2^*}+\mu f(x)w_\epsilon^{2^*})dx. \tag{2.9}$$

Next we claim that

$$\int_{\mathbb{R}^N}\triangle u w_\epsilon^{2^*}dx=\int_{\mathbb{R}^N}u\triangle w_\epsilon^{2^*}dx. \tag{2.10}$$

In fact, for any $R>0$, let $B(R)$ be a ball of radius R, we have

$$\int_{B(R)}\triangle u w_\epsilon^{2^*}dx-\int_{B(R)}u\triangle w_\epsilon^{2^*}dx=\int_{\partial B(R)}(\frac{\partial u}{\partial n}w_\epsilon^{2^*}-\frac{\partial w_\epsilon^{2^*}}{\partial n}u)dS$$

$$\leq w_\epsilon^{2^*}(R)\int_{\partial B(R)}|\nabla u|dS+|\nabla w_\epsilon^{2^*}|(R)\int_{\partial B(R)}|u|dS=O(R^{-2N})\int_{\partial B(R)}(|\nabla u|+|u|)dS$$

$$\leq O(R^{-2N})R^{\frac{N-1}{2}}\left[\left(\int_{\partial B(R)}|\nabla u|^2dS\right)^{\frac12}+\left(\int_{\partial B(R)}u^2dS\right)^{\frac12}\right]$$

$$=O(R^{-\frac{3N+1}{2}})\left[\left(\int_{\partial B(R)}|\nabla u|^2dS\right)^{\frac12}+\left(\int_{\partial B(R)}u^2dS\right)^{\frac12}\right].$$

Then the fact that $u \in H^1(\mathbb{R}^N)$ we see that the right hand side approaches 0 on a sequence of radii $R_i \longrightarrow \infty$.

¿From (2.9), (2.10) we get

$$\begin{aligned}\mu \int_{\mathbb{R}^N} f(x) w_\epsilon^{2^*} dx \le & -\int_{\mathbb{R}^N} u \triangle w_\epsilon^{2^*} dx - M \int_{\mathbb{R}^N} w_\epsilon^{2^*} u dx \\ & + C \int_{\mathbb{R}^N} w_\epsilon^{2^*} dx + \int_{\mathbb{R}^N} w_\epsilon^{2^*} u dx \\ & = C \int_{\mathbb{R}^N} w_\epsilon^{2^*} dx + \int_{\mathbb{R}^N} (1 - M - \frac{\triangle w_\epsilon^{2^*}}{w_\epsilon^{2^*}}) w_\epsilon^{2^*} u dx.\end{aligned} \tag{2.11}$$

¿From (2.1) we get

$$\begin{aligned}\frac{\triangle w_\epsilon^{2^*}}{w_\epsilon^{2^*}} &= \frac{\triangle (\epsilon + |x|^2)^{-N}}{(\epsilon + |x|^2)^{-N}} \\ &= 2N(N+2)(\epsilon + |x|^2)^{-2}(|x|^2 - \frac{N}{N+2}\epsilon) \\ &\ge 2N(N+2)(\epsilon + 0^2)^{-2}(0^2 - \frac{N}{N+2}\epsilon) = -2N^2\epsilon^{-1}.\end{aligned}$$

So (2.11) becomes

$$\mu \int_{\mathbb{R}^N} f(x) w_\epsilon^{2^*} dx \le C \int_{\mathbb{R}^N} w_\epsilon^{2^*} dx + (2N^2\epsilon^{-1} + 1 - M) \int_{\mathbb{R}^N} w_\epsilon^{2^*} u dx.$$

If we choose $M = 2N^2\epsilon^{-1} + 1$, then by using (2.3) we have

$$\mu \le \inf_{\epsilon > 0} \{ \frac{C \int_{\mathbb{R}^N} w_\epsilon^{2^*} dx}{\int_{\mathbb{R}^N} f(x) w_\epsilon^{2^*} dx} \} = \inf_{\epsilon > 0} \{ \frac{C S^{\frac{N}{2}}}{\int_{\mathbb{R}^N} f(x) w_\epsilon^{2^*} dx} \} = \mu_2.$$

In the following we give the expression of C. From (2.8), the constant C must satisfy

$$C \ge Mu - u^{2^*-1}. \tag{2.12}$$

Let $h(u) = Mu - u^{p-1}$ for $p = 2^* - 1$, it is easy to verify that $u = (\frac{M}{p-1})^{\frac{1}{p-2}}$ is the unique critical point which is a maximum of $h(u)$. From $h(0) = 0$ and $h(u) \longrightarrow -\infty$ as $u \longrightarrow +\infty$ we have

$$\sup_{u \ge 0} h(u) = h((\frac{M}{p-1})^{\frac{1}{p-2}})$$

So we can take

$$C = C_\epsilon = \sup_{u \ge 0} h(u) = (p-2)(\frac{M}{p-1})^{\frac{p-1}{p-2}} = \frac{4}{N-2} \left(\frac{(2N^2\epsilon^{-1} + 1)(N-2)}{(N+2)} \right)^{\frac{N+2}{4}}$$

then C satisfies (2.12) □

Proof of Theorem 1.

From Lemma 2.2 we set

$$\mu^* = \sup\{\mu > 0 \mid (1.1)_\mu - (1.2) \textit{ possesses at least one solution}\} \tag{2.13}$$

By Lemma 2.1 and Lemma 2.2 we have

$$0 < \mu_1 \leq \mu^* < \mu_2 < +\infty.$$

For any $\mu \in (0, \mu^*)$, by the definition of μ^* we can find a $\bar{\mu} \in (\mu, \mu^*)$ such that $(1.1)_{\bar{\mu}}$-(1.2) have a solution $u_{\bar{\mu}}$ and

$$-\triangle u_{\bar{\mu}} + u_{\bar{\mu}} - u_{\bar{\mu}}^{p-1} - \mu f(x) = (\bar{\mu} - \mu) f(x) \geq 0.$$

Thus $\tilde{u} = u_{\overline{\mu}}$ is a supersolution of $(1.1)_\mu$. ¿From $f(x) \geq 0$ we deduce that $\underset{\sim}{u} \equiv 0$ is a subsolution of $(1.1)_\mu$ for all $\mu > 0$. Again by the standard barrier method as in the proof of Lemma 2.1, we can obtain a solution u_μ of $(1.1)_\mu$ such that $0 \leq u_\mu \leq u_{\overline{\mu}}$. Since u_μ can be derived by an iteration scheme with initial value $u_{(0)} = 0$, u_μ is a minimal solution of $(1.1)_\mu$. Since 0 is not a solution of $(1.1)_\mu$, $\overline{\mu} > \mu$ and $f(x) \geq 0$, the maximum principle implies that

$$0 < u_\mu < u_{\overline{\mu}}. \tag{2.14}$$

Furthermore

$$\begin{aligned}
&\int_{\mathbb{R}^N} (|\nabla u_\mu|^2 + u_\mu^2)\, dx = \int_{\mathbb{R}^N} u_\mu^p \, dx + \mu \int_{\mathbb{R}^N} f(x) u_\mu \, dx \\
&\leq \int_{\mathbb{R}^N} u_{\overline{\mu}}^p \, dx + \overline{\mu} \int_{\mathbb{R}^N} f(x) u_{\overline{\mu}} \, dx = \int_{\mathbb{R}^N} (|\nabla u_{\overline{\mu}}|^2 + u_{\overline{\mu}}^2) dx < +\infty.
\end{aligned}$$

Thus $u_\mu \in H^1(\mathbb{R}^N)$.

By the definition of μ^* we can conclude that $(1.1)_\mu$-(1.2) have no solution for $\mu > \mu^*$. Therefore the proof of Theorem 1(i) and (iii) is completed.

Now we prove that $(1.1)_{\mu^*}$-(1.2) has a unique solution. Hence for the rest of this section we will assume that $p \in (2, \frac{2N}{N-2}]$. We shall use Lemma 2.3-2.5. The proof of Lemma 2.3-2.5 will be given later. From Lemma 2.3 we have

$$\int_{\mathbb{R}^N} (|\nabla u_\mu|^2 + u_\mu^2) dx - (p-1) \int_{\mathbb{R}^N} u_\mu^p dx > 0$$

and also we have

$$\int_{\mathbb{R}^N} (|\nabla u_\mu|^2 + u_\mu^2) dx - \int_{\mathbb{R}^N} u_\mu^p dx - \mu \int_{\mathbb{R}^N} f(x) u_\mu dx = 0$$

Thus

$$\begin{aligned}
\int_{\mathbb{R}^N} (|\nabla u_\mu|^2 + u_\mu^2) dx &= \int_{\mathbb{R}^N} u_\mu^p dx + \mu \int_{\mathbb{R}^N} f(x) u_\mu dx \\
&< \frac{1}{p-1} \int_{\mathbb{R}^N} (|\nabla u_\mu|^2 + u_\mu^2) dx + \mu^* (\int_{\mathbb{R}^N} f^2)^{\frac{1}{2}} (\int_{\mathbb{R}^N} u_\mu^2)^{\frac{1}{2}} \\
&\leq (\frac{1}{p-1} + \frac{\delta \mu^*}{2}) \|u_\mu\| + \frac{\mu^*}{2\delta} \int_{\mathbb{R}^N} f^2 dx
\end{aligned}$$

for any $\delta > 0$. Since $p > 2$ we can obtain that

$$\|u_\mu\|_{H^1(\mathbb{R}^N)} \leq C < +\infty$$

for all $\mu \in (0, \mu^*)$ by taking δ small enough. Because of (2.14) the solution u_μ is monotone increasing with respect to μ, we may suppose that

$$u_\mu \longrightarrow u_{\mu^*} \ \ \textit{weakly in } H^1(\mathbb{R}^N) \ \ \textit{as} \quad \mu \longrightarrow \mu^*$$

and hence u_{μ^*} is a solution of $(1.1)_{\mu^*}$ if $p \in (2, \frac{2N}{N-2}]$. The uniqueness of u_{μ^*} is obtained by Lemma 2.5. □

Lemma 2.3. *Let u_μ be the minimal positive solution given by Theorem 1(i). The corresponding eigenvalue problem*

$$\begin{cases} -\triangle\varphi + \varphi = \lambda(p-1)u_\mu^{p-2}\varphi, \\ \varphi \in H^1(\mathbb{R}^N) \end{cases} \tag{$(2.15)_\mu$}$$

has the first eigenvalue $\lambda_1 > 1$ and the corresponding eigenfunction $\varphi_1 > 0$ in $\mathbb{R}^N$ if $\mu \in (0, \mu^*)$.

Proof.

Set

$$\lambda_1 = \inf\{\int_{\mathbb{R}^N} (|\nabla v|^2 + v^2)dx \mid v \in H^1(\mathbb{R}^N), \ \int_{\mathbb{R}^N} (p-1)u_\mu^{p-2}v^2 = 1\}. \tag{2.16}$$

As in [27, 29], we can prove that the minimization (2.16) is achieved by some function $\varphi_1 > 0$. Thus (2.15) has a solution (λ_1, φ_1). Now we prove $\lambda_1 > 1$. For $\bar{\mu} \in (\mu, \mu^*)$ let u_μ, $u_{\bar{\mu}}$ be the minimal solutions of $(1.1)_\mu$-(1.2) and $(1.1)_{\bar{\mu}}$-(1.2) with $u_\mu < u_{\bar{\mu}}$. By virtue of $(1.1)_\mu$ and $(1.1)_{\bar{\mu}}$ we see that

$$-\triangle(u_{\bar{\mu}} - u_\mu) + (u_{\bar{\mu}} - u_\mu) = u_{\bar{\mu}}^{p-1} - u_\mu^{p-1} + (\bar{\mu} - \mu)f(x)$$

Applying the Taylor's expansion and noting that $\bar{\mu} > \mu$, $f(x) \geq 0$ we have

$$-\triangle(u_{\bar{\mu}} - u_\mu) + (u_{\bar{\mu}} - u_\mu) > (p-1)u_\mu^{p-2}(u_{\bar{\mu}} - u_\mu). \tag{2.17}$$

Multiplying (2.17) by φ_1 and using (2.15) we have

$$\lambda_1 \int_{\mathbb{R}^N} (p-1)u_\mu^{p-2}\varphi_1(u_{\bar{\mu}} - u_\mu)dx > \int_{\mathbb{R}^N} (p-1)u_\mu^{p-2}\varphi_1(u_{\bar{\mu}} - u_\mu)dx,$$

which gives $\lambda_1 > 1$. □

Lemma 2.4. *Assume that u_μ be a solution of $(1.1)_\mu$-(1.2) for which $\lambda_1 > 1$. Then for any $g(x) \in H^{-1}(\mathbb{R}^N)$, problem*

$$-\triangle w + w = (p-1)u_\mu^{p-2} w + g(x), \quad w \in H^1(\mathbb{R}^N) \qquad ((2.18)_\mu)$$

has a solution (here we suppose $u_0 \equiv 0$).

Proof.

Consider the functional

$$\Phi(w) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla w|^2 + w^2)dx - \frac{1}{2}(p-1)\int_{\mathbb{R}^N} u_\mu^{p-2} w^2 dx - \int_{\mathbb{R}^N} g(x) w dx, \quad w \in H^1(\mathbb{R}^N).$$

¿From Holder inequality and Young inequality we have, for any $\epsilon > 0$ that

$$\begin{aligned}\Phi(w) &\geq \frac{1}{2}(1 - \frac{1}{\lambda_1})\|w\|_{H^1(\mathbb{R}^N)} - \frac{1}{2}\epsilon\|w\|^2_{H^1(\mathbb{R}^N)} - \frac{C_\epsilon}{2}\|g\|^2_{H^{-1}(\mathbb{R}^N)} \\ &\geq -C\|g\|^2_{H^{-1}(\mathbb{R}^N)}\end{aligned} \qquad (2.19)$$

if we choose ϵ small.

Let $\{w_n\} \subset H^1(\mathbb{R}^N)$ be the minimizing sequence of variational problem

$$d = \inf\{\Phi(w) \mid w \in H^1(\mathbb{R}^N)\}.$$

¿From (2.19) we can also deduce that $\{w_n\}$ is bounded in $H^1(\mathbb{R}^N)$ if we choose ϵ small. So we may suppose that

$$w_n \longrightarrow w \text{ weakly in } H^1(\mathbb{R}^N) \text{ as } n \longrightarrow \infty,$$

$$w_n \longrightarrow w \text{ a.e. in } \mathbb{R}^N \text{ as } n \longrightarrow \infty.$$

By Fatou's Lemma

$$\|w\|^2_{H^1(\mathbb{R}^N)} \leq \underline{\lim}_{n \longrightarrow \infty} \|w_n\|^2_{H^1(\mathbb{R}^N)}.$$

The weak convergence and the fact that $u_\mu \longrightarrow 0$ *as* $x \longrightarrow \infty$ imply

$$\int_{\mathbb{R}^N} g w_n dx \longrightarrow \int_{\mathbb{R}^N} g w dx, \int_{\mathbb{R}^N} u_\mu^{p-2} w_n^2 dx \longrightarrow \int_{\mathbb{R}^N} u_\mu^{p-2} w^2 dx \text{ as } n \longrightarrow \infty.$$

Therefore

$$\Phi(w) \leq \lim_{n \longrightarrow \infty} \Phi(u_n) = d$$

and hence $\Phi(w) = d$ which gives that w is a solution of (4.20). □

Lemma 2.5. *Let $p \in (2, \frac{2N}{N-2}]$ and u_{μ^*} is a solution of $(1.1)_{\mu^*}$-(1.2), then problem $(2.15)_{\mu^*}$ has its first eigenvalue $\lambda_1(\mu^*) = 1$. Moreover, the solution u_{μ^*} is unique.*

Proof.

Define

$$F: \ \mathbb{R} \times H^1(\mathbb{R}^N) \longrightarrow H^{-1}(\mathbb{R}^N)$$

by

$$F(\mu, u) = \triangle u - u + (u^+)^{p-1} + \mu f(x).$$

Since $\lambda(\mu) > 1$ for $\mu \in (0, \mu^*)$, so $\lambda(\mu^*) \geq 1$. If $\lambda(\mu^*) > 1$, the equation $F_u(\mu^*, \ u_{\mu^*})\phi = 0$ has no nontrivial solution. From Lemma 2.4, F maps $\mathbb{R} \times H^1(\mathbb{R}^N)$ onto $H^{-1}(\mathbb{R}^N)$. Applying implicit function theorem to F we can find a neighborhood $(\mu^* - \delta, \mu^* + \delta)$ of μ^* such that $(1.1)_\mu$-(1.2) possesses a solution u_μ if $\mu \in (\mu^* + \delta, \ \mu^* + \delta)$. This is contradictory to the definition of μ^*.

Next, we are going to prove that u_{μ^*} is unique. In fact, if problem $(1.1)_{\mu^*}$-(1.2) has another solution $U_{\mu^*} \geq u_{\mu^*}$. Set $w = U_{\mu^*} - u_{\mu^*}$ we have

$$-\triangle w + w = (w + u_{\mu^*})^{p-1} - u_{\mu^*}^{p-1}, \ w > 0 \ in \ \mathbb{R}^N \tag{2.20}$$

By $\lambda_1(\mu^*) = 1$ we have that problem

$$-\triangle \phi + \phi = (p-1)u_{\mu^*}^{p-1}\phi, \quad \phi \in H^1(\mathbb{R}^N). \tag{2.21}$$

possesses a positive solution ϕ_1

Multiplying (2.20) by ϕ_1 and (2.21) by w, integrating and subtracting we deduce that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} [(w + u_{\mu^*})^{p-1} - u_{\mu^*}^{p-1} - (p-1)u_{\mu^*}^{p-2} w]\phi_1 dx \\ &= \frac{1}{2}(p-1)(p-2)\int_{\mathbb{R}^N} \xi_\mu^{p-3} w^2 \phi_1 dx \end{aligned}$$

where $\xi \in (u_{\mu^*}, \ u_{\mu^*} + w)$. Thus $w \equiv 0$. □

Remark 2.3. For $\mu \in (0, \mu^*)$, let u_μ be the minimal solution of $(1.1)_\mu$-(1.2). Set $\mu^{**} = \sup\{0 < \mu \leq \mu^*, \mid 1 - (p-1)u_\mu^{p-2} \geq 0\}$. From Remark 2.1 we have $\mu_1 \leq \mu^{**} \leq \mu^*$ and $1 - (p-1)u_\mu^{p-2} \geq 0$ for all $\mu \in (0, \mu^{**})$.

§3 Proof of Theorem 2

Let u_μ be the minimal positive solution of $(1.1)_\mu$-(1.2) for $\mu \in (0, \mu^*)$ and $p \in (2, \frac{2N}{N-2})$. In order to find a second solution of $(1.1)_\mu$-(1.2) we introduce the following problem:

$$\begin{cases} -\triangle v + v = (v + u_\mu)^{p-1} - u_\mu^{p-1}, \\ v \in H^1(\mathbb{R}^N), \quad v > 0 \ \ in \ \ \mathbb{R}^N. \end{cases} \tag{$(3.1)_\mu$}$$

Clearly, we can get another solution $U_\mu = u_\mu + v_\mu$ of $(1.1)_\mu$-(1.2) if $(3.1)_\mu$ possesses a positive solution v_μ. In this section, we prove that $(3.1)_\mu$ has a positive solution for

$\mu \in (0, \mu^*)$ by using variational method. To this end, We define the corresponding variational functional of $(3.1)_\mu$ by

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v|^2 + v^2)dx - \frac{1}{p}\int_{\mathbb{R}^N}[(v^+ + u_\mu)^p - u_\mu^p - pu_\mu^{p-1}v^+]dx, \tag{3.2}$$

with $v \in H^1(\mathbb{R}^N)$. For convenience, we use "$\|.\|$", "$|.|_q$" to denote the norms in $H^1(\mathbb{R}^N)$, $L^q(\mathbb{R}^N)$ respectively.

Let

$$I^\infty(v) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v|^2 + v^2)dx - \frac{1}{p}\int_{\mathbb{R}^N}|v|^p dx, \quad v \in H^1(\mathbb{R}^N), \tag{3.3}$$

$$M^\infty(v) = \{v \in H^1(\mathbb{R}^N), \ \ | \ \int_{\mathbb{R}^N}(|\nabla v|^2 + v^2)dx = \int_{\mathbb{R}^N}|v|^p dx\} \tag{3.4}$$

$$J^\infty = \inf\{I^\infty(v), \ \ | \ \ v \in M^\infty\}. \tag{3.5}$$

$$A = \inf\{\int_{\mathbb{R}^N}(|\nabla v|^2 + v^2)dx \ \ | \ \ v \in H^1(\mathbb{R}^N), \ \ |v|_p^p = 1\}$$

$$I_1(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u| + u^2)dx - \frac{1}{p}\int_{\mathbb{R}^N}(u^+)^p dx - \mu\int_{\mathbb{R}^N}f(x)u dx, \quad u \in H^1(\mathbb{R}^N)$$

and

$$\begin{aligned} I(v) = {} & \frac{1}{2}\int_{\mathbb{R}^N}[|\nabla(v + u_\mu)|^2 + (v + u_\mu)^2]dx \\ & - \frac{1}{p}\int_{\mathbb{R}^N}(v^+ + u_\mu)^p dx - \mu\int_{\mathbb{R}^N}f(x)(v + u_\mu)dx, \quad v \in H^1(\mathbb{R}^N). \end{aligned}$$

Because u_μ is the critical point of $I_1(u)$ we have that

$$J(v) = I(v) - I(0) = I(v) - I_1(u_\mu). \tag{3.6}$$

In the following we verify the existence of nontrivial solutions of problem $(3.1)_\mu$ by means of the Mountain Pass method.

Lemma 3.1. *There exists some constants $\alpha > 0, \ \rho > 0$ such that*

$$J(v)|_{\partial B_\rho} \geq \alpha > 0. \tag{3.7}$$

where $B_\rho = \{u \in H^1(\mathbb{R}^N) \ \ | \ \ \|u\| < \rho\}$

Proof.

For any $v \in H^1(\mathbb{R}^N)$, using Taylor's formula and Lemma 2.3 we have

$$\begin{aligned} J(v) \geq {} & \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v|^2 + v^2 - (p-1)u_\mu^{p-2}v^2)dx \\ & - \frac{\epsilon}{2}\int_{\mathbb{R}^N}v^2 dx - (C_\epsilon)\int_{\mathbb{R}^N}|v|^p dx \\ \geq {} & \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v|^2 + v^2)dx - \frac{1}{2\lambda_1}\int_{\mathbb{R}^N}(|\nabla v|^2 + v^2)dx \\ & - \frac{\epsilon}{2}\int_{\mathbb{R}^N}v^2 dx - (C_\epsilon)\int_{\mathbb{R}^N}|v|^p dx \\ = {} & (\frac{1}{2} - \frac{1}{2\lambda_1} - \frac{\epsilon}{2})\|v\|^2 - C_\epsilon\|v\|^p \end{aligned}$$

for all $\epsilon > 0$. Since $\lambda_1 > 1$ we can choose ϵ small enough so that $\frac{1}{2} - \frac{1}{2\lambda_1} - \frac{\epsilon}{2} > 0$. If we fix $\epsilon = \frac{1}{2}(\frac{\lambda_1 - 1}{\lambda_1})$, then $J(v) \geq \frac{\lambda_1 - 1}{4\lambda_1}\|v\|^2 - C_{\lambda_1}\|v\|^p$. Hence there exist $\rho > 0,\ \alpha > 0$ such that (3.7) holds. $\square$

Remark 3.1. $\rho > 0,\ \alpha > 0$ in Lemma 3.1 may depend on $\mu \in (0,\ \mu^*)$. But for any fixed $\delta \in (0,\ \mu^*)$, there exist $\rho_\delta,\ \alpha_\delta > 0$, independent of $\mu \in (0,\ \delta)$ such that

$$I(v) \geq \alpha_\delta > 0 \ if \ \|v\| = \rho_\delta$$

for all $\mu \in (0,\ \delta)$. Since from Remark 2.3 we find that u_μ is monotone increasing with respect to μ, so the first eigenvalue $\lambda_1(\mu)$ of $(2.15)_\mu$ must be nonincreasing in μ. Thus $\lambda_1(\mu) \geq \lambda_1(\delta)$ if $\mu \in (0,\ \delta]$. The conclusion follows by the same argument as in the proof of Lemma 3.1 using $\lambda_1(\delta)$ there.

Lemma 3.2. *For any $0 \leq v \in H^1(\mathbb{R}^N),\ v \not\equiv 0$, there exists a constant $R_0 > 0$ such that*

$$J(Rv) \leq 0 \quad for \quad R \geq R_0. \tag{3.8}$$

Proof.

From the inequality ([14])

$$(v + u_\mu)^p - u_\mu^p - v^p \geq p u_\mu^{p-1} v \quad for \ \ any \ \ v \geq 0, \quad p \geq 2 \tag{3.9}$$

we have, for $v \in H^1(\mathbb{R}^N),\ v \not\equiv 0,\ R \in \mathbb{R}^+$, that

$$\begin{aligned} J(Rv) &\leq \frac{1}{2}R^2 \int_{\mathbb{R}^N} (|\nabla v|^2 + v^2)dx - \frac{R^p}{p}\int_{\mathbb{R}^N} (v^+)^p dx \\ &\leq \frac{1}{2}R^2\|v\|^2 - (\frac{1}{p})R^p|v|_p^p. \end{aligned}$$

we deduce that $I(Rv) \longrightarrow -\infty \ \ as \ \ R \longrightarrow +\infty$. $\square$

Theorem 3.1. *Let $p \in (2, 2^*)$, $\mu \in (0, \mu^*)$. If there exists $0 \leq v_0 \in H^1(\mathbb{R}^N)$ with $v_0 \not\equiv 0$ such that*

$$\sup_{t \geq 0} J(tv_0) < J^\infty, \tag{3.10}$$

then $(3.1)_\mu$ has at least one positive solution.

Proof.

By Lemma 3.1 and Lemma 3.2, there exists a constant $R_1 > 0$ such that $e = R_1 v_0 \notin B_\rho$ and $J(e) \leq 0$. Where $B_\rho = \{u \in H^1(\mathbb{R}^N) \mid \|u\| < \rho\}$. Define $c = \inf_{\Gamma \in \mathcal{D}} \max_{v \in \Gamma} J(v)$, where $\mathcal{D}$ denote the class of continuous paths joining 0 to e in $H^1(\mathbb{R}^N)$. We have

$$0 < \alpha \leq c \leq \sup_{t \geq 0} J(tv_0) < J^\infty \tag{3.11}$$

¿From Mountain Pass Lemma without $(PS)_c$ condition [7, 10] we can find a sequence $\{v_j\} \subset H^1(\mathbb{R}^N)$ such that

$$J(v_j) \longrightarrow c \quad as \quad j \longrightarrow \quad \infty, \tag{3.12}$$

$$J'(v_j) \longrightarrow 0 \quad in \quad (H^1(\mathbb{R}^N))^* \quad as \quad j \longrightarrow \infty. \tag{3.13}$$

Using (3.12) and (3.13) together with $p > 2$ we can easily prove that $\{v_j\}$ is bounded in $H^1(\mathbb{R}^N)$. By passing to a subsequence if necessary we may assume that for some $d > 0$

$$v_j \longrightarrow v \quad weakly \;\; in \;\; H^1(\mathbb{R}^N),$$

$$v_j \longrightarrow v \quad ae. \;\; in \quad R^N \;\; and \; \|v_j\| \longrightarrow d \;\; since \;\; c > 0,$$

and

$$(v_j^+ + u_\mu)^{p-1} - u_\mu^{p-1} \longrightarrow (v^+ + u_\mu)^{p-1} - u_\mu^{p-1} \; weakly \;\; in \;\; (L^p(\mathbb{R}^N))^*.$$

Thus v is a weak solution of

$$-\triangle v + v = (v^+ + u_\mu)^{p-1} - u_\mu^{p-1}. \tag{3.14}$$

Using maximum principle we get $v \geq 0$ in $\mathbb{R}^N$. Set $u_j = v_j + u_\mu$, $u = v + u_\mu$ then

$$u_j \longrightarrow u \quad weakly \;\; in \;\; H^1(\mathbb{R}^N),$$

$$u_j \longrightarrow u \quad ae. \;\; in \;\; \mathbb{R}^N$$

By (3.6) (3.12) (3.13) and (3.14) we have

$$\begin{cases} I_1(u_j) - I_1(u_\mu) = I(v_j) - I(0) = J(v_j) \longrightarrow c > 0 \\ I_1'(u_j) \longrightarrow 0 \quad in \quad (H^1(\mathbb{R}^N))^* \end{cases} \tag{3.15}$$

as $j \longrightarrow \infty$, and u is a solution of

$$-\triangle u + u = u^{p-1} + \mu f(x). \tag{3.16}$$

Now we are going to prove that $u \not\equiv u_\mu$. In fact, if $u \equiv u_\mu$, then $v \equiv 0$ and $v_j \not\longrightarrow 0$ strongly in $H^1(\mathbb{R}^N)$ since $J(0) = 0 < c$. Let $c_1 = c + I_1(u_\mu)$. From the fact that $u_j \longrightarrow u_\mu$ weakly in $H^1(\mathbb{R}^N)$ and Brezis-Lieb Lemma[6] we have

$$\begin{cases} \|u_j\|^2 = \|u_\mu\|^2 + \|v_j\|^2 + o(1), \\ |u_j^+|^p = |u_\mu|^p + |v_j^+|^p + o(1), \\ \displaystyle\int_{\mathbb{R}^N} f(x) u_j dx = \int_{\mathbb{R}^N} f(x) u_\mu dx + o(1) \end{cases} \tag{3.17}$$

as $j \longrightarrow \infty$. By (3.15) (3.16) we deduce

$$\int_{\mathbb{R}^N} (|\nabla u_j|^2 + u_j^2) dx = \int_{\mathbb{R}^N} (u_j^+)^p dx + \mu \int_{\mathbb{R}^N} f(x) u_j dx + o(1),$$

$$\int_{\mathbb{R}^N} (|\nabla u_\mu|^2 + u_\mu^2) dx = \int_{\mathbb{R}^N} (u_\mu^+)^p dx + \mu \int_{\mathbb{R}^N} f(x) u_\mu dx$$

and hence
$$\int_{\mathbb{R}^N}(|\nabla v_j|^2+v_j^2)dx=\int_{\mathbb{R}^N}(v_j^+)^p dx+o(1), \tag{3.18}$$
by subtracting the two identities above and by (3.17). Using (3.15), (3.16) (3.17) and (3.18) we deduce that, as $j\longrightarrow\infty$,
$$\begin{aligned} c_1 &= I_1(u_j)+o(1)\\ &= I_1(u_\mu)+\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v_j|^2+v_j^2)dx-\frac{1}{p}\int_{\mathbb{R}^N}(v_j^+)^p dx+o(1)\\ &= I_1(u_\mu)+(\frac{1}{2}-\frac{1}{p})\int_{\mathbb{R}^N}(v_j^+)^p dx+o(1). \end{aligned} \tag{3.19}$$
Similarly we obtain that
$$A|v_j^+|_p^2\le \|v_j\|_{H^1(\mathbb{R}^N)}^2=|v_j^+|_p^p+o(1)$$
which gives us that $\|v_j^+\|_p^p+o(1)\ge A^{\frac{p}{p-2}}$ since $d>0$. Thus
$$c+I_1(u_\mu)=c_1\ge I_1(u_\mu)+(\frac{1}{2}-\frac{1}{p})A^{\frac{p}{p-2}} \tag{3.20}$$
Using the results of [4] [23] we obtained that both J^∞ and A can been attained by some functions w and v respectively and both w and $A^{\frac{1}{p-2}}v$ are the ground state solutions. By the uniqueness of ground state solution of $(1.1)_0$ [20, 21, 24] we have $w=A^{\frac{1}{p-2}}v$ and hence
$$J^\infty=I^\infty(w)=I^\infty(A^{\frac{1}{p-2}}v)=(\frac{1}{2}-\frac{1}{p})A^{\frac{p}{p-2}}.$$
From (3.20) we deduce $c\ge J^\infty$, which is a contradiction because of (3.11). □

Theorem 3.2. *Let u_μ be the minimal positive solution of $(1.1)_\mu$ with $\mu\in(0,\mu^*)$. Then $(3.1)_\mu$ possesses at least one positive solution v_μ with $0<J(v_\mu)<J^\infty$ if $p\in(2,2^*)$.*

Proof.

Let $w(x)$ be a minimizer of J^∞ then there exists a $t^*>0$ such that $\sup\{J(tw),\ \ t>0\}=J(t^*w)$ and by the fact that $w(x)$ is the unique ground state solution of $(1.1)_0$ we have
$$\begin{aligned} J(t^*w) &= \frac{(t^*)^2}{2}\int_{\mathbb{R}^N}(|\nabla w|^2+w^2)dx-\frac{1}{p}\int_{\mathbb{R}^N}((t^*w+u_\mu)^p-u_\mu^p-pu_\mu^{p-1}t^*w)dx\\ &= \frac{(t^*)^2}{2}\int_{\mathbb{R}^N}(|\nabla w|^2+w^2)dx-\frac{1}{p}\int_{\mathbb{R}^N}(t^*)^p w^p dx\\ &-\frac{1}{p}\int_{\mathbb{R}^N}((t^*w+u_\mu)^p-u_\mu^p-pu_\mu^{p-1}t^*w-(t^*)^p w^p)dx\\ &= I^\infty(t^*w)-\frac{1}{p}\int_{\mathbb{R}^N}((t^*w+u_\mu)^p-u_\mu^p-pu_\mu^{p-1}t^*w-(t^*)^p w^p)dx\\ &\le \sup_{t>0}\{I^\infty(tw)\}-\frac{1}{p}\int_{\mathbb{R}^N}((t^*w+u_\mu)^p-u_\mu^p-pu_\mu^{p-1}t^*w-(t^*)^p w^p)dx\\ &\le J^\infty-\frac{1}{p}\int_{\mathbb{R}^N}((t^*w+u_\mu)^p-u_\mu^p-pu_\mu^{p-1}t^*w-(t^*)^p w^p)dx\\ &< J^\infty \quad \textit{since } (3.9). \end{aligned}$$

Using theorem 3.1 we get our result. □

Proof of Theorem 2.

From Theorem 1, $(1.1)_\mu$ possesses a minimal positive solution u_μ if $\mu \in (0, \mu^*]$. we use Theorem 3.2 to get the solution v_μ for $(3.1)_\mu$ if $\mu \in (0, \mu^*)$. Taking $U_\mu = u_\mu + v_\mu$, then $U_\mu > u_\mu$ is another positive solution of $(1.1)_\mu$. □

§4. Proof of Theorem 3

Lemma 4.1. *Let U_0 be the ground state solution of $(1.1)_0$, then for any $g(x) \in H^{-1}(\mathbb{R}^N)$ problem*

$$-\triangle w + w = (p-1)U_0^{p-2}w + g(x),\ w \in H^1(\mathbb{R}^N) \tag{4.1}$$

has a solution if

$$-\triangle \phi + \phi = (p-1)U_0^{p-2}\phi, \quad \phi \in H^1(\mathbb{R}^N) \tag{4.2}$$

has no nontrivial solution.

Proof.

Same as Lemma 2.4 we can prove that problem

$$-\triangle u + u = h(x),\ u \in H^1(\mathbb{R}^N)$$

has a unique solution for all $h(x) \in H^{-1}(\mathbb{R}^N)$. Thus the operator $-\triangle + 1$ is a isomorphism from $H^1(\mathbb{R}^N)$ onto $H^{-1}(\mathbb{R}^N)$ and the inverse $(-\triangle + 1)^{-1}$ exist. To prove (4.1) has a solution, it is sufficient to prove that

$$w = (-\triangle + 1)^{-1}U_0^{p-2}w + (-\triangle + 1)^{-1}g(x),\ w \in H^1(\mathbb{R}^N) \tag{4.3}$$

has a solution. Set $T = (-\triangle + 1)^{-1}U_0^{p-2}$, $g_1(x) = (-\triangle + 1)^{-1}g(x)$. Then (4.3) becomes

$$w - Tw = g_1(x), \quad w \in H^1(\mathbb{R}^N). \tag{4.4}$$

We claim that T is a compact operator from $H^1(\mathbb{R}^N)$ to $H^1(\mathbb{R}^N)$. In fact, Let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^N)$ and suppose that

$$u_n \longrightarrow u \textit{ weakly in } H^1(\mathbb{R}^N) \textit{ as } n \longrightarrow \infty.$$

Let $w_n = Tu_n$. Then

$$-\triangle w_n + w_n = (p-1)U_0^{p-2}u_n \tag{4.5}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N}(|\nabla w_n|^2 + w_n^2)dx = (p-1)\int_{\mathbb{R}^N} U_0^{p-2}u_n w_n &\le \max U_0^{p-2}|w_n|_2|u_n|_2 \\ &\le \frac{\epsilon}{2}\|w_n\|^2 + \frac{C_\epsilon}{2}(p-1)\max U_0^{p-2}\|u_n\|^2\end{aligned}$$

for any $\epsilon > 0$. Taking $\epsilon = \frac{1}{2}$, we deduce that $\{w_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Thus we may suppose that

$$w_n \longrightarrow w \textit{ weakly in } H^1(\mathbb{R}^N) \textit{ as } n \longrightarrow \infty.$$

and hence

$$-\triangle w + w = (p-1)U_0^{p-2}u. \tag{4.6}$$

¿From (4.5) (4.6) we deduce

$$-\triangle(w_n - w) + (w_n - w) = (p-1)U_0^{p-2}(u_n - u)$$

and

$$\int_{\mathbb{R}^N}(|\nabla(w_n-w)|^2 + (w_n-w)^2)dx = (p-1)\int_{\mathbb{R}^N} U_0^{p-2}(u_n-u)(w_n-w)dx$$
$$\leq (p-1)[\int_{\mathbb{R}^N} U_0^{p-2}(u_n-u)^2dx]^{\frac{1}{2}}[\int_{\mathbb{R}^N} U_0^{p-2}(w_n-w)^2dx]^{\frac{1}{2}}.$$

Because U_0^{p-2} is the ground state of $(1.1)_0$, so there exists a constant $C > 0$ such that $|U_0^{p-2}| \leq C$. By Young inequality we have

$$(1 - \frac{\epsilon}{2}\|U_0\|_{L^\infty}^{p-2})\int_{\mathbb{R}^N}(|\nabla(w_n-w)|^2 + (w_n-w)^2)dx \leq \frac{C_\epsilon}{2}(p-1)\int_{\mathbb{R}^N} U_0^{p-2}(u_n-u)^2dx$$

for all $\epsilon > 0$. Taking ϵ small enough we deduce

$$\int_{\mathbb{R}^N} |\nabla(w_n-w)|^2 + (w_n-w)^2dx \leq C\int_{\mathbb{R}^N} U_0^{p-2}(u_n-u)^2dx$$
$$= C\int_{|x|\leq R} U_0^{p-2}(u_n-u)^2dx + C\int_{|x|\geq R} U_0^{p-2}(u_n-u)^2dx$$

Using the exponential delay of U_0^{p-2} we have that, for any $\epsilon > 0$, there exists an $R > 0$ such that $U_0^{p-2}(x) < \epsilon$ for all $|x| > R$. Thus

$$\|w_n - w\|^2 \leq C\int_{|x|\leq R}(u_n-u)^2dx + C\epsilon. \tag{4.7}$$

By the compact Sobolev embedding, it follows that for $n \geq n_0,\ n_0 > 1$ large, that

$$\int_{|x|\leq R}(u_n-u)^2dx < \epsilon. \tag{4.8}$$

Thus (4.7) (4.8) and the arbitrariness of ϵ imply that

$$w_n \longrightarrow w \textit{ strongly in } H^1(\mathbb{R}^N) \textit{ as } n \longrightarrow \infty.$$

By the Fredholm alternative theorem we deduce our Lemma. □

Applying the results of [28] we have the following Lemma.

Lemma 4.2. *Let $U_0(r) = U_0(|x|)$ be the unique ground state solution of* $(1.1)_0$ *and* $\lambda_1(U_0),\ \lambda_2(U_0)$ *are the first and second eigenvalue of*

$$\begin{cases} -\delta'' - \dfrac{N-1}{r}\delta' + \delta - \lambda(p-1)U_0^{p-2}\delta = 0 \\ \delta(\infty) = 0, \quad \delta'(0) = 0 \end{cases} \tag{4.9}$$

respectively. Then $\lambda_1(U_0) < 1,\ \lambda_2(U_0) > 1.$

Proof of Theorem 3. We define

$$F:\ \mathbb{R}^1 \times H^1(\mathbb{R}^N) \longrightarrow H^{-1}(\mathbb{R}^N)$$

by

$$F(\mu,\ u) = \triangle u - u + |u|^{p-2}u + \mu f(x),\ u \in H^1(\mathbb{R}^N),\ \mu \in \mathbb{R}^1. \tag{4.10}$$

It can be verified that F is well-defined and differentiable. Let $U_0(x) = U_0(r)$ be the unique positive solution of $(1.1)_0^*$. From Lemma 4.2 and notice that

$$F_u(0,\ U_0)\delta = \triangle\delta - \delta + (p-1)U_0^{p-2}\delta,\ \delta \in H^1(\mathbb{R}^N).$$

we have that $F_u(0, U_0)\delta = 0$ has no nontrivial solutions. We refer by Lemma 4.1 and implicit function theorem that the solution of $F(\mu,\ u) = 0$ near $(0, U_0)$ are given by a continuous curve $(\mu,\ U(\mu))$ with $U(0) = U_0$. Thus there exists a constant $\mu_* > 0$ such that problem $(1.1)_\mu^*$ has a solution U_μ if $\mu \in (-\mu_*,\ \mu_*)$ and $U_\mu \longrightarrow U_0$ as $\mu \longrightarrow 0$ in $H^1(\mathbb{R}^N)$. Notice that $-u(-\mu,\ x))$ must be the solution of $(1.1)_\mu^*$ if $u(-\mu,\ x)$ is a solution of $(1.1)_{-\mu}^*$ we deduce that $\hat{U}_\mu = -U(-\mu,\ x)$ is the third solution of $(1.1)_\mu^*$ with $\mu \in (0,\ \mu_*)$. □

Remark 4.1. For $\mu < 0$, we can also get three solutions for problem $(1.1)_\mu^*$ if $\mu \in (-\mu_*,\ 0)$ since the solutions of $(1.1)_\mu^*$ are odd with respect to μ.

§5 Proof of Theorem 4

Lemma 5.1. *Let $f(x) \in C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. Then $U \subset L^\infty(\mathbb{R}^N)$ and U is uniformly bounded in $L^\infty(\mathbb{R}^N)$ if $p \in (2,\ \frac{2N}{N-2})$, where U is given by (1.6).*

Proof.

By elliptic regular theory [18] we can deduce that $U \subset C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$. Suppose on the contrary when $p \in (2,\ \frac{2N}{N-2})$ that there is a sequence $\{u_n\} \subset U$ such that $\sup_{x\in\mathbb{R}^N} u_n \longrightarrow +\infty$. Take

$$M_n = \sup_{x\in\mathbb{R}^N} u_n(x) = u_n(x_n), \quad y = \alpha x + x_n, \quad w_n(x) = \frac{1}{M_n}u_n(\alpha x + x_n). \tag{5.1}$$

Where α is some constant to be determined latter. Clearly, $0 \le w_n(x) \le 1$ and $w_n(0) = 1$. Because u_n are the solutions of $(1.1)_\mu$, we have

$$-M_n\frac{1}{\alpha^2}\triangle w_n(x) + M_n w_n(x) = M_n^{p-1}w_n^{p-1}(x) + \mu_n f(\alpha x + x_n). \tag{5.2}$$

Letting $\alpha = M_n^{\frac{p-2}{2}}$ we have

$$-\triangle w_n(x) + M_n^{2-p} w_n(x) = w_n^{p-1}(x) + \frac{\mu_n}{M_n^{p-1}} f(\alpha x + x_n). \tag{5.3}$$

¿From $0 \leq w_n(x) \leq 1$ and the elliptic regular theory we deduce that $w_n(x)$ is bounded in $C^{2,\alpha}(\mathbb{R}^N)$. So we can suppose that

$$w_n(x) \longrightarrow w_\infty \ in \ C^2(\mathbb{R}^N) \ as \ n \longrightarrow \infty,$$

and hence w_∞ is a nontrivial positive solution of

$$-\triangle w = w^{p-1} \ with \ \lim_{|x| \longrightarrow \infty} w(x) = 0 \ and \ w(0) = 1,$$

which is impossible [8, 11, 17]. □

Lemma 5.2. *Let $f(x) \in C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, Then for any $g(x) \in C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ problem* $(2.18)_\mu$ *has a solution $w \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ for all $\mu \in (0,\ \mu^*)$ (again we suppose here $u_0 \equiv 0$).*

Proof.

From Lemma 2.4 we know that $(2.18)_\mu$ has a solution $w \in H^1(\mathbb{R}^N)$. By the assumption on f and g, it is known from [26; Proposition 4.3] that $w \in H^2(\mathbb{R}^N)$. The standard elliptic regular theory yields $w \in C^{2,\alpha}(\mathbb{R}^N)$ □

Similarly, we can prove that

Lemma 5.3. *Let $f(x) \in C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, then for any $g(x) \in C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ problem (4.1) has a solution $w \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$.*

Proof of Theorem 4. The conclusion i) comes immediately from Lemma 5.1. As for ii) we define

$$F : \mathbb{R}^1 \times C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \longrightarrow C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \tag{5.4}$$

by

$$F(\mu,\ u) = \triangle u - u + (u^+)^{p-1} + \mu f(x). \tag{5.5}$$

Where $C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ and $C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ are endowed with the natural norms, then they become a Banach spaces. It can be verified that $F(\mu,\ u)$ is differentiable. From Lemma 2.3 and Lemma 5.3 we know that for $\mu \in (0, \mu^*)$

$$F_u(\mu,\ u_\mu)w = \triangle w - w + (p-1)u_\mu^{p-2} w$$

is an isomorphism of $C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ onto $C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. It follows from Implicit Function Theorem that the solutions of $F(\mu,\ u) = 0$ near $(\mu,\ u_\mu)$ are given by a continuous curve.

Now we are going to prove that $(\mu^*,\ u_{\mu^*})$ is a bifurcation point in $C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ by using an idea in [30]. To this end, we needs the following bifurcation theorem [12]:

Theorem A. *Let X, Y be Banach space. Let $(\bar{\lambda}, \bar{x}) \in \mathbb{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood of $(\bar{\lambda},\ \bar{x})$ into Y. Let the null-space $N(F_x(\bar{\lambda},\ \bar{x})) = span\{x_0\}$ be one-dimensional and $codim R(F_x(\bar{\lambda},\ \bar{x})) = 1$. Let $F_\lambda(\bar{\lambda},\ \bar{x}) \notin R(F_x(\bar{\lambda},\ \bar{x}))$. If Z is the complement of $span\{x_0\}$ in X, then the solution of $F(\lambda,\ x) = F(\bar{\lambda},\ \bar{x})$ near $(\bar{\lambda},\ \bar{x})$ form a curve $(\lambda(s),\ x(s)) = (\bar{\lambda} + \tau(s),\ \bar{x} + s x_0 + z(s))$. Where $s \longrightarrow (\tau(s),\ z(s)) \in \mathbb{R} \times Z$ is continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = z(0) = z'(0) = 0$.*

Define F as (5.4) (5.5). We show that at the critical point $(\mu^*,\ u_{\mu^*})$, the Theorem A applies. Indeed, from Lemma 2.5, problem (2.21) has a solution $\phi_1 > 0$ in $\mathbb{R}^N$. $\phi_1 \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ if $f \in C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. Thus $F(\mu^*,\ u_{\mu^*})\phi = 0$, $\phi \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ has a solution $\phi_1 > 0$. This implies that $N(F_u(\mu^*,\ u_{\mu^*})) = span\{\phi_1\} = 1$ is one dimensional and $codim R(F_u(\mu^*,\ u_{\mu^*})) = 1$ by the Fredholm alternative. It remains to check that $F_\mu(\mu^*,\ u_{\mu^*}) \notin R(F_u(\mu^*,\ u_{\mu^*}))$.

Assuming the contrary would imply existence of $v(x) \not\equiv 0$ such that

$$\triangle v - v + (p-1)u_{\mu^*}^{p-2} v = -f(x), \quad v \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$$

¿From $F_u(\mu^*,\ u_{\mu^*})\phi_1 = 0$ we conclude that $\int_{\mathbb{R}^N} f(x)\phi_1 dx = 0$. This is impossible because $f(x) \geq 0$, $f(x) \not\equiv 0$ and $\phi_1(x) > 0$ in $\mathbb{R}^N$.

Applying Theorem A we conclude that $(\mu^*,\ u_{\mu^*})$ is the bifurcation point near which, the solution of $(1.1)_\mu$-(1.2) form a curve $(\mu^* + \tau(s),\ u_{\mu^*} + s\phi_1 + z(s))$ with s near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$. We claim that $\tau''(0) < 0$ which implies that the bifurcation curve turns strictly to the left in $(\mu,\ u)$ plane. Since $\mu = \mu^* + \tau(s)$, $u = u_{\mu^*} + s\phi_1 + z(s)$ in

$$-\triangle u + u - u^{p-1} - \mu f(x) = 0,\ u > 0, \quad u \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N). \tag{5.6}$$

Differentiate (5.6) in s twice we have

$$-\triangle u_{ss} + u_{ss} - (p-1)(p-2)u^{p-3}u_s^2 - (p-1)u^{p-2}u_{ss} - \mu_{ss} f(x) = 0$$

Set here $s = 0$ and use that $\tau'(0) = 0$, $u_s = \phi_1(x)$ and $u = u_{\mu^*}$ as $s = 0$ we obtain

$$-\triangle u_{ss} + u_{ss} - (p-1)(p-2)u_{\mu^*}^{p-3}\phi_1^2 - (p-1)u_{\mu^*}^{p-2}u_{ss} + \tau''(0) f(x) = 0 \tag{5.7}$$

Multiplying

$$F_u(\mu^*,\ u_{\mu^*})\phi_1 = 0$$

by u_{ss}, and (5.7) by ϕ_1, integrating and subtracting the result we obtain

$$(p-1)(p-2)\int_{\mathbb{R}^N} u_{\mu^*}^{p-3}\phi_1^3 dx + \tau''(0)\int_{\mathbb{R}^N} f(x) dx = 0$$

which immediately give $\tau''(0) < 0$.

Thus

$$u_\mu \longrightarrow u_{\mu^*} \quad in \quad C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)\ as\ \mu \longrightarrow \mu^*,$$

$$U_\mu \longrightarrow u_{\mu^*} \ \ in \ \ C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \ as \ \mu \longrightarrow \mu^*.$$

Using Lemma 5.1, Lemma 5.2, Lemma 5.3, Lemma 4.2, implicit function theorem and the uniqueness of positive ground state solution of $(1.1)_0$ we can easily prove that

$$u_\mu \longrightarrow 0 \ in \ C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \ as \ \mu \longrightarrow 0,$$

and

$$U_\mu \longrightarrow U_0 \ in \ C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \ as \ \mu \longrightarrow 0.$$

□

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