Multiple solutions and bifurcation of nonhomogeneous semilinear elliptic equations in $\mathbb{R}^N$ *

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1 Introduction

In our recent paper [DLZ], we considered the existence of two solutions for nonhomogeneous problem

$$\begin{cases} -\triangle u + u = f(x,u) + \mu h(x), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1$_\mu$}$$

with $h(x) \in L^\infty(\mathbb{R}^N) \cap H^{-1}(\mathbb{R}^N)$ and $\mu > 0$. Under the assumptions of

f$_1$) $f(x,u) \in C^1((0,\infty), \mathbb{R}^1)$ with respect to u;

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f$_2$) there exist $C_1 > 0$, $C_2 \in (0,1)$ such that $|f(x,t)| \leq C_1 |t|^{p-1} + C_2 t$ for $x \in \mathbb{R}^N$, $t \in (0,\infty)$ and $\lim_{t\to\infty} \frac{f(x,t)}{t} = +\infty$ uniformly for $x \in \mathbb{R}^N$ where $2 < p < +\infty$ and $N \geq 2$;

f$_3$) there exists a constant $\alpha \in (0,1)$ such that $\alpha t f_t'(x,t) \geq f(x,t) \geq 0$ for all $x \in \mathbb{R}^N$ $t \in (0,\ \infty)$.

and

h) $h(x) \in L^\infty(\mathbb{R}^N) \cap H^{-1}(\mathbb{R}^N)$, $h(x) \geq 0$, $h(x) \not\equiv 0$ in $\mathbb{R}^N$ and $\lim_{|x|\to\infty} h(x) = 0$.

we have got the following result:

Theorem 1.1. *If* f$_1$), f$_2$), f$_3$) *and h) hold, then there exists a positive constant* $\mu^* < +\infty$ *such that problem* $(1.1)_\mu$ *has at least one minimal positive solution* u_μ *if* $\mu \in (0,\mu^*)$ *and there are no solutions for* $(1.1)_\mu$ *if* $\mu > \mu^*$*; furthermore,* u_μ *is increasing with respect to* $\mu \in (0,\mu^*]$ *and there is a unique solution for* $(1.1)_{\mu^*}$ *if* $p < \frac{2N}{N-2}$ *when* $N \geq 3$.

Defining the variational functional of $(1.1)_\mu$ by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right)\, dx - \int_{\mathbb{R}^N} F(x,u)\, dx - \mu \int_{\mathbb{R}^N} h(x)\, u\, dx,$$

where $F(x,u) = \int_0^u f(x,t)\, dt$, we also have the next theorem.

Theorem 1.2. *If in addition to* f$_2$), *and* f$_3$) *with* $p < \frac{2N}{N-2}$ *if* $N \geq 3$ *we suppose*

f$_4$) $f(x,\cdot) \in C^2(0,+\infty)$, $\frac{\partial^2 f}{\partial t^2} \geq 0$ *for* $x \in \mathbb{R}^N$, $t > 0$,

f$_5$) $\lim_{t\to 0^+} t \cdot \frac{\partial^2 f}{\partial t^2} = 0$ *uniformly for* $x \in \mathbb{R}^N$, $t \geq 0$, $\lim_{t\to\infty} t^{1-q} \left|\frac{\partial^2 f}{\partial t^2}\right| \leq C$ *uniformly for* $x \in \mathbb{R}^N$ *where* $C > 0$ *is some constant and* $0 < q < \frac{4}{N-2}$,

f$_6$) $\lim_{|x|\to\infty} f(x,t) = \bar{f}(t)$ *uniformly for bounded* $t > 0$ *and* $f(x,t) \geq \bar{f}(t)$ *for all* $x \in \mathbb{R}^N$,

then problem $(1.1)_\mu$ *has at least two positive solutions* u_μ, U_μ *with* $u_\mu < U_\mu$ *if* $\mu \in (0,\mu^*)$ *and* u_μ *is a local minimiser of* $I(u)$.

In this paper, we will continue to discuss the existence and bifurcation of multiple solutions for problem $(1.1)_\mu$ for the subcritical case. We will also give a result about the uniqueness of the positive solution of problem $(1.1)_\mu$ for the critical case and supercritical case. For simplicity, we suppose that $f(x,\ u) = f(u)$ to be independent of x throughtout this paper. More precisely, we will consider the inhomogeneous elliptic problem:

$$\begin{cases} -\triangle u + u = f(u) + \mu h(x) & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.2$_\mu$}$$

under the assumptions:

I) $f(t) \in C^1(-\infty,\ \infty)$ with $f'(0) \in [0,\ 1)$; $f(t) \not\equiv 0$; $f(t)$ is odd and there exists a constant $\alpha \in (0,1)$ such that $\alpha t^2 f'(t) \geq f(t)\, t \geq 0$ for all $t \in (-\infty,\ \infty)$.

II) $f(t) \in C^2(0, +\infty)$, $f''(t) \geq 0$ for all $t \geq 0$ and $\lim_{t\to 0^+} t f''(t) = 0$, $\lim_{t\to\infty} t^{1-q} f''(t) \leq C$ for some constants $C > 0$ and $0 < q < \frac{4}{N-2}$ if $N \geq 3$; $0 < q < \infty$ if $N = 2$.

III) There exists a positive constant B such that

$$\frac{f(u)}{u^{p-1}} \to B \quad as \quad u \to \infty,$$

where $2 < p < +\infty$ if $N = 2$ and $2 < p < \frac{2N}{N-2}$ if $N \geq 3$.

IV) For any given $\beta_0 > \beta$ there exists a constant $C_0 = C_0(\beta_0) > 1$ such that

$$C_0(f(u) - u) - (f'(u) - 1)u \leq 0, \quad \text{for } 0 < u < \beta_0$$
$$C_0(f(u) - u) - (f'(u) - 1)u \geq 0, \quad \text{for } u > \beta_0$$

where β is the unique zero point of $f(u) - u$ in $(0,\ \infty)$.

Our main results are as follows.

Theorem 1.3. *Suppose I) − IV) and h). Then there exists a positive constant* $\mu_* < \mu^*$ *such that* $(1.2)_\mu$ *has at least three solutions if* $\mu \in (0, \mu_*)$, *and two of them are positive.*

Theorem 1.4. *Suppose I)–IV), and h), moreover,* $h(x) \in C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. *Then*

(i) *The set of solutions*

$$U = \{u \in H^1(\mathbb{R}^N) \;\mid\; u \text{ is a solution of } (1.2)_\mu\} \tag{1.3}$$

is bounded uniformly in $L^\infty(\mathbb{R}^N)$.

(ii) u_μ *is continuous and increasing with respect to* μ *if* $\mu \in (0,\ \mu^*)$ *for all* $x \in \mathbb{R}^N$.

(iii) $(\mu^*,\ u_{\mu^*})$ *is in* $C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ *and is a bifurcation point for* $(1.2)_\mu$ *and*

$$u_\mu \longrightarrow 0 \;\; in \;\; C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \;\; as \;\; \mu \longrightarrow 0,$$

$$U_\mu \longrightarrow U_0 \;\; in \;\; C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \;\; as \;\; \mu \longrightarrow 0.$$

Where U_0 *is the unique positive solution of* $(1.2)_0$, u_μ *is the minimal solution of* $(1.2)_\mu$ *and* U_μ *is the second solution of* $(1.2)_\mu$ *constructed in Theorem 1.2.*

(iv) *For any* $\delta > 0$, *there exist* $C, R > 0$ *such that*

$$U_\mu(x) - u_\mu(x) \le C \exp\left(-\left(1 - f'(0) - \delta\right)|x|\right) \qquad for \;\; |x| \ge R.$$

However when h satisfies the following (1.4) we obtain the following uniqueness result.

Theorem 1.5. *Let* $N \ge 6$, $h(x) \in L^2(\mathbb{R}^N) \cap C^\alpha(\mathbb{R}^N)$, $\frac{4}{N-2} f'(t) - u f''(t) \le 0$ *and* $f''(t)$ *is nonincreasing and suppose additionally that*

$$\frac{\partial h}{\partial x_i} x_i \le 0 \;\; and \;\; h(x_1, ..., x_i, ..., x_N) = h(x_1, ..., -x_i, ..., x_N) \;\; for \;\; i = 1,\ 2,\ ...,\ N. \tag{1.4}$$

Then there exists a constant $\mu_{**} > 0$ *such that* $(1.1)_\mu$ *has only one solution, namely the minimal solution if* $\mu \in (0, \mu_{**})$.

Corollary 1.6. *Suppose* $N \ge 6$, $\frac{2N}{N-2} \le p_i \le 3$ *for* $i = 1, 2, ..., m$ *and (1.4). Then problem* $(1.1)_\mu$ *has a unique positive solution if* $f(u) = \sum_{i=1}^m c_i u^{p_i}$ *and* μ *is small, where* c_i *are some positive constants.*

Remark 1.7. If we assume that the conditions $I)$, $II)$ hold, the condition $IV)$ is equivalent to the following condition (see [KZ], [CD]):

$IV)'$ The function $G(u) = \frac{u(f'(u)-1)}{f(u)-u}$ is nonincreasing in $(\beta, +\infty)$ and converges to a finite limit $G^* \ge 1$ as $u \to +\infty$; $G(u) \le G^*$ in $[0, \beta]$.

Remark 1.8. From the assumptions $I)$, $II)$ and $IV)$, we can conclude the uniqueness of positive solution of $(1.2)_0$ (see [KZ], [CD]).

Remark 1.9. By using the assumptions $I)$, $II)$ and $III)$ we can easily deduce the assumptions $f_1) - f_5$ with $p = q + 2$, $C_1 = B + \epsilon$, $C_2 = f'(0)$.

We shall organize this paper as follows. In §2 some preliminary results are given including the study of the linearized eigenvalue problems associated with the minimum solutions described in Theorem 1.1. In §3 we prove Theorem 1.3 by the implicit function theorem. In §4 we prove Theorem 1.4 by a bifurcation theorem [CR]. Finally, we prove Theorem 1.5 by using an improved Pohozaev identity and moving plan technique in §5.

2 Preliminary Lemmas

In this section, we give some Lemmas which will be used in the proof of our main results.

Lemma 2.1. *If $I)$–$II)$ and $h)$ hold, then the minimization problem*

$$\lambda_1 = \inf \left\{ \int_{\mathbb{R}^N} |\nabla v|^2 + (1 - f'(0)) v^2 \,\middle|\, v \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} (f'(u_\mu) - f'(0)) v^2 \, dx = 1 \right\} \quad (2.1)$$

can be achieved by some $v_0 > 0$, and furthermore, $\lambda_1 > 1$, where u_μ is the minimal solution of $(1.2)_\mu$ with $\mu \in (0, \mu^*)$.

Proof. Notice that $f'(0) \in [0,1)$, $\int_{\mathbb{R}^N} |\nabla v|^2 + (1 - f'(0)) v^2 \geq (1 - f'(0)) \|v\|^2_{H^1(\mathbb{R}^N)}$. It is easy to see that $\lambda_1 < +\infty$. Let $\{v_n\} \subset H^1(\mathbb{R}^N)$ be a minimizing sequence of λ_1, that is,

$$\int_{\mathbb{R}^N} (f'(u_\mu) - f'(0)) v_n^2 dx = 1, \qquad \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + (1 - f'(0))\right) v_n^2 dx \longrightarrow \lambda_1$$

and $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Without loss of generality (at least by choosing a subsequence) we can assume, for some $v_0 \in H^1(\mathbb{R}^N)$, that

$$v_n \longrightarrow v_0 \quad \text{weakly in } H^1(\mathbb{R}^N) \quad \text{as } n \longrightarrow \infty,$$
$$v_n \longrightarrow v_0 \quad \text{a.e. in } \mathbb{R}^N \quad \text{as } n \longrightarrow \infty.$$

Hence

$$\int_{\mathbb{R}^N} |\nabla v_0|^2 + (1 - f'(0)) v_0^2 dx \leq \underline{\lim}_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_0|^2 + (1 - f'(0)) v_n^2\, dx = \lambda_1. \tag{2.2}$$

To prove that v_0 achieves λ_1, it suffices to show that

$$\int_{\mathbb{R}^N} (f'(u_\mu) - f'(0)) v_0^2 = 1. \tag{2.3}$$

For this purpose, we need some estimates of $f(t)$. By II), for any $\varepsilon > 0$ we can find $C_\varepsilon > 0$ such that

$$f''(t) \leq \varepsilon t^{-1} + C_\varepsilon t^{q-1} \quad t > 0, \tag{2.4}$$

where $0 < q < \frac{4}{N-2}$. Consequently, there is a constant $C > 0$ such that

$$f'(t) \leq Ct^q \qquad \text{for } x \in \mathbb{R}^N,\ t > 1. \tag{2.5}$$

From (2.4) and (2.5) we deduce that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f'(t) - f'(0)| < \varepsilon + C_\varepsilon t^q \qquad \text{for all } t > 0 \tag{2.6}$$

For any fixed $R > 0$, let $B_R = \left\{x \in \mathbb{R}^N \mid |x| < R\right\}$. We have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |f'(u_\mu) - f'(0)|\, |v_n - v_0|^2\, dx \\
&\quad\leq \int_{B_R} |f'(u_\mu) - f'(0)|\, |v_n - v_0|^2\, dx \\
&\qquad + \int_{\mathbb{R}^N\setminus B_R} |f'(u_\mu) - f'(0)|\, |v_n - v_0|^2\, dx \\
&\quad\leq \int_{B_R} \left(\varepsilon + C_\varepsilon u_\mu^q\right) |v_n - v_0|^2\, dx + \int_{\mathbb{R}^N\setminus B_R} \left(\varepsilon + C_\varepsilon u_\mu^q\right) |v_n - v_0|^2\, dx \\
&\quad\leq \varepsilon \int_{\mathbb{R}^N} |v_n - v_0|^2\, dx + C_\varepsilon \left[\left(\int_{B_R} u_\mu^{q+2} dx\right)^{\frac{q}{q+2}} \left(\int_{B_R} |v_n - v_0|^{q+2}\, dx\right)^{\frac{2}{q+2}}\right] \\
&\qquad + C_\varepsilon \left[\left(\int_{\mathbb{R}^N\setminus B_R} u_\mu^{q+2} dx\right)^{\frac{q}{q+2}} \left(\int_{\mathbb{R}^N\setminus B_R} |v_n - v_0|^{q+2}\, dx\right)^{\frac{2}{q+2}}\right].
\end{aligned} \tag{2.7}$$

Since $v_n \to v_0$ strongly in $L^s(B_R)$ for $2 \leq s < \frac{2N}{N-2}$, $\{v_n\}$ is a bounded sequence in $H^1\left(\mathbb{R}^N\right)$. Taking $n \to \infty$, then $R \to \infty$ and finally $\varepsilon \to 0^+$ we deduce (2.3). Therefore,

v_0 achieves λ_1. Clearly $|v_0|$ also achieves λ_1. Hence we may assume $v_0 \geq 0$ in $\mathbb{R}^N$ and v_0 satisfies

$$-\triangle v_0 + (1 - f'(0)) v_0 = \lambda_1 (f'(u_\mu) - f'(0)) v_0. \tag{2.8}$$

Once again, by the maximum principle for weak solutions we deduce that $v_0 > 0$ in $\mathbb{R}^N$.

We will now prove that $\lambda_1 > 1$. By the definition of u_μ we obtain for any $\mu_1 < \mu_2$

$$\begin{aligned} -\triangle (u_{\mu_2} - u_{\mu_1}) + (u_{\mu_2} - u_{\mu_1}) &= f(u_{\mu_2}) - f(u_{\mu_1}) + (\mu_2 - \mu_1) h(x) \\ &\geq f'(u_{\mu_1})(u_{\mu_2} - u_{\mu_1}) + (\mu_2 - \mu_1) h(x). \end{aligned} \tag{2.9}$$

Multiplying (2.9) by v_0 and integrating it over $\mathbb{R}^N$, we get

$$\int_{\mathbb{R}^N} \nabla (u_{\mu_2} - u_{\mu_1}) \nabla v_0 + (u_{\mu_2} - u_{\mu_1}) v_0 \, dx > \int_{\mathbb{R}^N} f'(u_{\mu_1})(u_{\mu_2} - u_{\mu_1}) \, dx. \tag{2.10}$$

By (2.8) we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \nabla (u_{\mu_2} - u_{\mu_1}) \nabla v_0 + (u_{\mu_2} - u_{\mu_1}) v_0 dx \\ &\quad = \lambda_1 \int_{\mathbb{R}^N} (f'(u_{\mu_2}) - f'(0))(u_{\mu_2} - u_{\mu_1}) v_0 dx + \int_{\mathbb{R}^N} f'(0)(u_{\mu_2} - u_{\mu_1}) v_0 dx. \end{aligned} \tag{2.11}$$

By (2.10) and (2.11) we deduce that

$$\lambda_1 \int_{\mathbb{R}^N} (f'(u_{\mu_2}) - f'(0))(u_{\mu_2} - u_{\mu_1}) v_0 dx > \int_{\mathbb{R}^N} (f'(u_{\mu_2}) - f'(0))(u_{\mu_2} - u_{\mu_1}) v_0 dx,$$

which implies that $\lambda_1 > 1$.

By the definition of λ_1 we have

$$\int_{\mathbb{R}^N} |\nabla v|^2 + (1 - f'(0)) v^2 \, dx \geq \lambda_1 \int_{\mathbb{R}^N} (f'(u_\mu) - f'(0)) v^2 \, dx \tag{2.12}$$

for all $v \in H^1(\mathbb{R}^N)$. □

Lemma 2.2. *Suppose I), II) and h). Assume that u_μ is a solution of $(1.2)_\mu$ for which $\lambda_1 > 1$ as defined in (2.1). Then for any $g(x) \in H^{-1}(\mathbb{R}^N)$, the problem*

$$-\Delta w + w = f'(u_\mu) w + g(x) \qquad w \in H^1(\mathbb{R}^N) \tag{2.13$_\mu$}$$

has a solution (here we suppose $u_0 \equiv 0$).

Proof. Consider the functional

$$\Phi(w) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla w|^2 + w^2\right) dx - \frac{1}{2}\int_{\mathbb{R}^N} f'(u_\mu) w^2 \, dx - \int g(x) w \, dx$$

$w \in H^1(\mathbb{R}^N)$. From (2.12), Holder's inequality and Young's inequality we have

$$\begin{aligned}
\Phi(w) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla w|^2 + (1 - f'(0)) w^2\right) dx \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^N} (f'(u_\mu) - f'(0)) w^2 \, dx - \int g(x) w \, dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla w|^2 + (1 - f'(0)) w^2\right) dx \\
&\quad - \frac{1}{2\lambda_1}\int_{\mathbb{R}^N} \left(|\nabla w|^2 + (1 - f'(0))\right) w^2 \, dx - \int g(x) w \, dx \\
&\geq \left(\frac{1}{2} - \frac{1}{2\lambda_1}\right)\int_{\mathbb{R}^N} |\nabla w|^2 + (1 - f'(0)) w^2 \, dx - \frac{\varepsilon}{2}\|w\|_{H^1}^2 - \frac{C_\varepsilon}{2}\|g\|_{H^{-1}}^2 .
\end{aligned}$$

By $I)$ we have $1 - f'(0) > 0$ Thus

$$\begin{aligned}
\Phi(w) &\geq \left(\frac{1}{2} - \frac{1}{2\lambda_1}\right)\int_{\mathbb{R}^N} |\nabla w|^2 + (1 - f'(0)) w^2 \, dx - \frac{\varepsilon}{2}\|w\|_{H^1}^2 - \frac{C_\varepsilon}{2}\|g\|_{H^{-1}}^2 \qquad (2.14) \\
&\geq \left[\frac{1}{2}\left(1 - \frac{1}{\lambda_1}\right)(1 - f'(0)) - \frac{\varepsilon}{2}\right]\|w\|_{H^1}^2 - \frac{C_\varepsilon}{2}\|g\|_{H^{-1}}^2 \\
&\geq -C\|g\|_{H^{-1}}^2
\end{aligned}$$

if we choose ε small.

Let $\{w_n\} \subset H^1(\mathbb{R}^N)$ be the minimizing sequence of the variational problem

$$d = \inf\left\{\Phi(w) \mid w \in H^1(\mathbb{R}^N)\right\}.$$

From (2.14) we have

$$\begin{aligned}
\left[\frac{1}{2}\left(1 - \frac{1}{\lambda_1}\right)(1 - f'(0)) - \frac{\varepsilon}{2}\right]\|w_n\|_{H^1}^2 &\leq \Phi(w_n) + \frac{C_\varepsilon}{2}\|g\|_{H^{-1}}^2 \\
&\leq d + \frac{C_\varepsilon}{2}\|g\|_{H^{-1}}^2 + o(1) \qquad \text{as } n \longrightarrow \infty.
\end{aligned}$$

By $\lambda_1 > 1$ and $f'(0) \in (0,1)$ we deduce that $\{w_n\}$ is bounded in $H^1(\mathbb{R}^N)$ if we choose ε small. So we may suppose that

$$\begin{aligned}
w_n &\longrightarrow w \qquad \text{weakly in } H^1(\mathbb{R}^N) \qquad \text{as } n \longrightarrow \infty, \\
w_n &\longrightarrow w \qquad \text{a.e. in } \mathbb{R}^N \qquad \text{as } n \longrightarrow \infty.
\end{aligned}$$

By Fatou's lemma

$$\|w\|_{H^1(\mathbb{R}^N)}^2 \leq \underline{\lim}_{n\to\infty} \|w_n\|_{H^1(\mathbb{R}^N)}^2 .$$

We now prove that

$$\int_{\mathbb{R}^N} \left(f'\left(u_\mu\right) - f'\left(0\right)\right) \left(w_n - w\right)^2 \, dx \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty. \tag{2.15}$$

In fact, by (2.6), for any $\varepsilon > 0$, $R > 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left|f'\left(u_\mu\right) - f'\left(0\right)\right| \left|w_n - w\right|^2 \, dx \\
&\quad \leq \int_{B_R(0)} \left|f'\left(u_\mu\right) - f'\left(0\right)\right| \left|w_n - w\right|^2 \, dx \\
&\qquad + \int_{\mathbb{R}^N\setminus B(0)} \left|f'\left(u_\mu\right) - f'\left(0\right)\right| \left|w_n - w\right|^2 \, dx \\
&\quad \leq \int_{B_R(0)} \left(\varepsilon + C_\varepsilon u_\mu^q\right) \left|w_n - w\right|^2 \, dx + \int_{\mathbb{R}^N\setminus B_R} \left(\varepsilon + C_\varepsilon u_\mu^q\right) \left|w_n - w\right|^2 \, dx \\
&\quad \leq \varepsilon \int_{\mathbb{R}^N} \left|w_n - w\right|^2 \, dx + C_\varepsilon \left[\left(\int_{B_R} u_\mu^{q+2} dx \right)^{\frac{q}{q+2}} \left(\int_{B_R} \left|w_n - w\right|^{q+2} dx \right)^{\frac{2}{q+2}} \right] \\
&\qquad + C_\varepsilon \left[\left(\int_{\mathbb{R}^N\setminus B_R} u_\mu^{q+2} dx \right)^{\frac{q}{q+2}} \left(\int_{\mathbb{R}^N\setminus B_R} \left|w_n - w\right|^{q+2} dx \right)^{\frac{2}{q+2}} \right].
\end{aligned}$$

Since $w_n \to w$ strongly in $L^s\left(B_R\right)$ for $2 \leq s < \frac{2N}{N-2}$, and $\{w_n\}$ is a bounded sequence in $H^1\left(\mathbb{R}^N\right)$, taking $n \to \infty$, then $R \to \infty$, and finally $\varepsilon \to 0^+$, we deduce our claim.

From (2.15) and the definition of weak convergence we can easily deduce that

$$\int_{\mathbb{R}^N} \left(f'\left(u_\mu\right) - f'\left(0\right)\right) w_n^2 \, dx \longrightarrow \int_{\mathbb{R}^N} \left(f'\left(u_\mu\right) - f'\left(0\right)\right) w^2 \, dx$$

and

$$\int g w_n \longrightarrow \int g w$$

as $n \to \infty$. Thus

$$\begin{aligned}
\Phi(w) &= \frac{1}{2}\int |\nabla w|^2 + w^2\,dx - \frac{1}{2}\int f'(u_\mu)\,w^2\,dx - \int g(x)\,w\,dx \\
&\le \frac{1}{2}\underline{\lim}_{n\to\infty}\int |\nabla w|^2 + w^2\,dx - \frac{1}{2}\lim_{n\to\infty}\int f'(u_\mu)\,w_n^2\,dx \\
&\quad - \lim_{n\to\infty}\int g(x)\,w_n\,dx \\
&= \underline{\lim}_{n\to\infty}\Phi(w_n) = d = \inf_{w\in H^1}\Phi(w),
\end{aligned}$$

and hence

$$\Phi(w) = d,$$

which implies that w is a solution of $(2.13)_\mu$. □

Lemma 2.3. *Suppose $I) - II)$ and $h)$, let u_{μ^*} be a solution of $(1.2)_{\mu^*}$ Then problem $(1.2)_{\mu^*}$ has its first eigenvalue $\lambda_1(\mu^*) = 1$.*

Proof. Define

$$F\colon \mathbb{R}\times H^1\left(\mathbb{R}^N\right) \longrightarrow H^{-1}\left(\mathbb{R}^N\right)$$

by

$$F(\mu, u) = \triangle u - u + f\left(u^+\right) + \mu h(x).$$

Since $\lambda_1(\mu) > 1$ for $\mu \in (0, \mu^*)$, it follows that $\lambda_1(\mu^*) \ge 1$. If $\lambda_1(\mu^*) > 1$, the equation $F_u(\mu^*, u_{\mu^*})\,\phi = 0$ has no nontrivial solution. From Lemma 2.2, F maps $\mathbb{R}\times H^1\left(\mathbb{R}^N\right)$ onto $H^{-1}\left(\mathbb{R}^N\right)$. Applying the implicit function theorem to F we can find a neighborhood $(\mu^* - \delta, \mu^* + \delta)$ of μ^* such that $(1.2)_\mu$ possesses a solution u_μ if $\mu \in (\mu^* - \delta, \mu^* + \delta)$. This is contradictory to the definition of μ^*.

□

3 The existence of the third solution

In this section, we will prove the existence of the third solution for problem $(1.2)_\mu$ by the implicit function theorem.

The following Lemma is a composition of the results in [GNN], [BL], [KZ] and [CD].

Lemma 3.1. *Suppose I), II) and IV). Let $U_0(x)$ be a ground state solution of $(1.2)_0$. Then U_0 is unique, radially symmtric, decreasing and*

$$\lim_{|x|\to\infty} U_0(|x|)e^{m|x|}|x|^{\frac{N-1}{2}} = C > 0,$$

where $m^2 = 1 - f'(0)$. Moreover, the corresponding linearized problem

$$\begin{cases} -\delta'' - \frac{N-1}{r}\delta' + \delta - f'(U_0)\,\delta = 0, \\ \delta(\infty) = 0, \qquad \delta'(0) = 0, \end{cases}$$

has no nontrivial solution.

Lemma 3.2. *Let U_0 be the ground state solution of $(1.2)_0$. Then for any $g(x) \in H^{-1}(\mathbb{R}^N)$ the problem*

$$-\triangle w + w = f'(U_0)\,w + g(x), \qquad w \in H^1(\mathbb{R}^N), \tag{3.1}$$

has a solution if

$$-\triangle \phi + \phi = f'(U_0)\,\phi, \qquad \phi \in H^1(\mathbb{R}^N), \tag{3.2}$$

has no nontrivial solution.

Proof. In the same way as in Lemma 2.2, we can prove that the problem

$$-\triangle u + u = h(x), \qquad u \in H^1(\mathbb{R}^N),$$

has a unique solution for all $h(x) \in H^{-1}(\mathbb{R}^N)$. Thus the operator $-\triangle +1$ is an isomorphism from $H^1(\mathbb{R}^N)$ onto $H^{-1}(\mathbb{R}^N)$ and the inverse $(-\triangle+1)^{-1}$ exists. To prove that (3.1) has a solution, it is sufficient to prove that

$$w = (-\triangle+1)^{-1} f'(U_0)\,w + (-\triangle+1)^{-1} g(x), \qquad w \in H^1(\mathbb{R}^N), \tag{3.3}$$

has a solution. Set

$$T = (-\triangle+1)^{-1} f'(U_0), \qquad g_1(x) = (-\triangle+1)^{-1} g(x).$$

Then (3.3) becomes

$$w - Tw = g_1(x), \qquad w \in H^1(\mathbb{R}^N). \tag{3.4}$$

We claim that T is a compact operator from $H^1(\mathbb{R}^N)$ to $H^1(\mathbb{R}^N)$. In fact, let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^N)$ and suppose that

$$u_n \longrightarrow u \qquad \text{weakly in } H^1(\mathbb{R}^N) \text{ as } n \longrightarrow \infty.$$

Let $w_n = Tu_n$. Then

$$-\triangle w_n + w_n = f'(U_0)\, u_n, \tag{3.5}$$

and from (2.4),

$$\begin{aligned} \int_{\mathbb{R}^N} \left(|\nabla w_n|^2 + w_n^2\right) dx &= \int_{\mathbb{R}^N} f'(U_0)\, u_n w_n \\ &\le \max\left(\varepsilon \ln U_0 + \frac{1}{q} U_0^q\right) |w_n|_2\, |u_n|_2 \\ &\le \frac{\varepsilon}{2} \|w_n\|^2 + \frac{C_\varepsilon}{2} \|u_n\|^2 \end{aligned}$$

for any $\varepsilon > 0$. Taking $\varepsilon = \frac{1}{2}$, we deduce that $\{w_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Thus we may suppose that

$$w_n \longrightarrow w \qquad \text{weakly in } H^1(\mathbb{R}^N) \text{ as } n \longrightarrow \infty,$$

and hence

$$-\triangle w + w = f'(U_0)\, u. \tag{3.6}$$

From (3.5)–(3.6) we deduce

$$-\triangle (w_n - w) + (w_n - w) = f'(U_0)(u_n - u)$$

and

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(|\nabla (w_n - w)|^2 + (w_n - w)^2\right) dx \\ &\quad = \int_{\mathbb{R}^N} f'(U_0)(u_n - u)(w_n - w)\, dx \\ &\quad \le \left[\int_{\mathbb{R}^N} f'(U_0)(u_n - u)^2\, dx\right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^N} f'(U_0)(w_n - w)^2\, dx\right]^{\frac{1}{2}}. \end{aligned}$$

Because U_0 is the ground state of $(1.2)_0$, there exists a constant $C > 0$ such that $|U_0| \le C$. By Young's inequality we have

$$\left(1 - \frac{\varepsilon}{2} \|f(U_0)\|_{L^\infty}\right) \int_{\mathbb{R}^N} \left(|\nabla (w_n - w)|^2 + (w_n - w)^2\right) dx \le \frac{C_\varepsilon}{2} \int_{\mathbb{R}^N} f'(U_0)(u_n - u)^2 \, dx$$

for all $\varepsilon > 0$. Taking ε small enough we deduce

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla (w_n - w)|^2 + (w_n - w)^2 \, dx &\le C \int_{\mathbb{R}^N} f'(U_0)(u_n - u)^2 \, dx \\ &= C \int_{|x| \le R} f'(U_0)(u_n - u)^2 \, dx + C \int_{|x| \ge R} f'(U_0)(u_n - u)^2 \, dx. \end{aligned}$$

Using the exponential decay of $f'(U_0)$ we have that, for any $\varepsilon > 0$, there exists an $R > 0$ such that $f'(U_0(x)) < \varepsilon$ for all $|x| > R$. Thus

$$\|w_n - w\|^2 \le C \int_{|x| \le R} (u_n - u)^2 \, dx + C\varepsilon. \tag{3.7}$$

By the compact Sobolev embedding, it follows, for $n \ge n_0$, $n_0 > 1$ large, that

$$\int_{|x| \le R} (u_n - u)^2 \, dx < \varepsilon. \tag{3.8}$$

Thus (3.7)–(3.8) and the arbitrariness of ε imply that

$$w_n \longrightarrow w \qquad \text{strongly in } H^1(\mathbb{R}^N) \text{ as } n \longrightarrow \infty.$$

By the Fredholm alternative theorem we deduce our lemma. □

Proof of Theorem 1.3. We define

$$F \colon \mathbb{R}^1 \times H^1(\mathbb{R}^N) \longrightarrow H^{-1}(\mathbb{R}^N)$$

by

$$F(\mu, u) = \triangle u - u + f(u) + \mu h(x), \qquad u \in H^1(\mathbb{R}^N), \ \mu \in \mathbb{R}^1. \tag{3.9}$$

It can be verified that F is well-defined and differentiable. Let $U_0(x) = U_0(r)$ be the unique positive solution of $(1.2)_0$. From Lemma 3.1, noticing that

$$F_u(0, U_0)\,\delta = \triangle\delta - \delta + f'(U_0)\,\delta, \qquad \delta \in H^1\left(\mathbb{R}^N\right),$$

we have that $F_u(0, U_0)\,\delta = 0$ has no nontrivial solution. We infer by Lemma 3.2 and the implicit function theorem that the solutions of $F(\mu, u) = 0$ near $(0, U_0)$ are given by a continuous curve $(\mu, U(\mu))$ with $U(0) = U_0$. Thus there exists a constant $\mu_* > 0$ such that problem $(1.2)_\mu$ has a solution U_μ if $\mu \in (-\mu_*, \mu_*)$, and $U_\mu \to U_0$ as $\mu \to 0$ in $H^1\left(\mathbb{R}^N\right)$. Noticing that $(-u(-\mu, x))$ must be a solution of $(1.2)_\mu$ if $u(-\mu, x)$ is a solution of $(1.2)_{-\mu}$, we deduce that $\hat{U}_\mu = -U(-\mu, x)$ is the third solution of $(1.2)_\mu$ with $\mu \in (0, \mu_*)$. □

Remark 3.3. For $\mu < 0$, we can also get three solutions for problem $(1.2)_\mu$ if $\mu \in (-\mu_*, 0)$, since the solutions of $(1.2)_\mu$ are odd with respect to μ.

4 Propositions and bifurcation

Proposition 4.1. *Suppose I)-III) and h). If $u \in H^1\left(\mathbb{R}^N\right)$, $u > 0$, is a solution of $(1.2)_\mu$, then*

(i) *$u(x)$ and $|\nabla u(x)|$ go to zero uniformly as $|x| \to \infty$;*

(ii) *for any $\varepsilon > 0$, there is a constant $C > 0$ such that*

$$u(x) \geq C \exp\left(\left(-(1+\varepsilon)\right) C\,|x|\right), \qquad |x| \geq R,$$

for $R > 0$ large enough.

Proof. We adapt the argument by H. Bresis and T. Kato [BK] to deduce that $u \in L^q\left(\mathbb{R}^N\right)$ for q large.

Letting $i > 1$, multiplying $(1.2)_\mu$ by u^i and integrating by parts we obtain

$$4i\,(1+i)^{-2} \int_{\mathbb{R}^N} \left|\nabla u^{\frac{1}{2}(1+i)}\right|^2 dx + \int_{\mathbb{R}^N} u^{1+i}\,dx = \int_{\mathbb{R}^N} f(u)\,u^i\,dx + \mu \int h(x)\,u^i\,dx.$$

Because of remark 1.9 and f_2), we have

$$\begin{aligned}\int_{\mathbb{R}^N} f(u)\,u^i\,dx &\le \int_{\mathbb{R}^N}\left(C_1 u^{p-1}+f'(0)u\right)u^i\,dx\\ &= C_1\int_{\mathbb{R}^N} u^{p-1}u^i\,dx + f'(0)\int_{\mathbb{R}^N} u^{1+i}\,dx\\ &\le C_1\int_{\mathbb{R}^N} u^{p-1+i}\,dx + f'(0)\left(C\int_{\mathbb{R}^N}\left(u^{p-1+i}+u^2\right)\,dx\right)\\ &\le C_3\int_{\mathbb{R}^N} u^{p-1+i}\,dx + C_4\int_{\mathbb{R}^N} u^2\,dx.\\ \int_{\mathbb{R}^N} h(x)\,u^i\,dx &\le C\left(\int_{\mathbb{R}^N} h^{\frac{p+i}{p}}(x)\,dx\right)^{\frac{p}{p+i}}\left(\int_{\mathbb{R}^N} u^{p+i}\,dx\right)^{\frac{i}{p+i}}\\ &\le C_1\left(\int_{\mathbb{R}^N} h^{\frac{p+i}{p}}(x)\,dx\right) + f'(0)\int_{\mathbb{R}^N} u^{p+i}\,dx.\end{aligned}$$

Thus

$$\begin{aligned}4i\,(1+i)^{-2}\int_{\mathbb{R}^N}\left|\nabla u^{\frac{1}{2}(1+i)}\right|^2\,dx &\le \int_{\mathbb{R}^N} f(u)\,u^i\,dx + \mu^*\int_{\mathbb{R}^N} h(x)\,u^i\,dx \qquad (4.1)\\ &\le C\int_{\mathfrak{N}}\left(u^{p+i}+u^2\right)\,dx + C\int_{\mathbb{R}^N} h^{\frac{p+i}{p}}(x)\,dx\\ &\le C\int_{\mathbb{R}^N}\left(u^{p+i}+u^2\right)\,dx + C \qquad (4.2)\end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Then for $i \ge \frac{2N}{N-2} - (p-1)$ we have

$$t^{p-1+i} \le \varepsilon t^{i+\frac{N+2}{N-2}} + C_\varepsilon t^{\frac{2N}{N-2}} \qquad \text{for } t \ge 0 \qquad (4.3)$$

(because $\frac{2N}{N-2} \le (p-1)+i < i+\frac{N+2}{N-2}$) by Young's inequality. Applying the Sobolev's inequality we find

$$\begin{aligned}\left(\int_{\mathbb{R}^N} u^q\,dx\right)^{\frac{N-2}{N}} &\le \left(\int_{\mathbb{R}^N} \left(u^{\frac{1}{2}(i+1)}\right)^{\frac{2N}{N-2}}\,dx\right)^{\frac{N-2}{N}}\\ &\le C\int_{\mathbb{R}^N}\left|\nabla u^{\frac{1}{2}(1+i)}\right|^2 dx\\ &\le C\left(\int_{\mathbb{R}^N}\left[u^{\frac{(N-2)q}{N}+\frac{4}{N-2}} + u^{\frac{2N}{N-2}} + u^2\right] dx + C\right)\\ &\le C\left(\int_{\mathbb{R}^N} u^q\,dx\right)^{\frac{N-2}{N}}\left(\int_{\mathbb{R}^N} u^{\frac{2N}{N-2}}\,dx\right)^{\frac{2}{N}} + C,\end{aligned}$$

since $u \in H^1\left(\mathbb{R}^N\right)$, where $q = \frac{N(1+i)}{N-2}$. Again by Young's inequality we have, for any $\varepsilon > 0$,

$$\begin{aligned}\int_{\mathbb{R}^N} u^q\,dx &\le C\left(\int_{\mathbb{R}^N} u^q\,dx\right)^{\frac{N-2}{N}}\left(\int_{\mathbb{R}^N} u^{\frac{2N}{N-2}}\,dx\right)^{\frac{2}{N}}\\ &\le \frac{\varepsilon C}{2}\int_{\mathbb{R}^N} u^q\,dx + \frac{C}{2\varepsilon}\int_{\mathbb{R}^N} u^{\frac{2N}{N-2}}\,dx.\end{aligned}$$

Taking $\varepsilon > 0$ small enough, we have

$$\begin{aligned}\left(1-\frac{C\varepsilon}{2}\right)\int_{\mathbb{R}^N} u^q\,dx &\le \frac{C}{2\varepsilon}\int_{\mathbb{R}^N} u^{\frac{2N}{N-2}}\,dx\\ \int_{\mathbb{R}^N} u^q\,dx &\le \left(1-\frac{C\varepsilon}{2}\right)^{-1}\cdot\frac{C}{2\varepsilon}\int_{\mathbb{R}^N} u^{\frac{2N}{N-2}}\,dx \le C.\end{aligned}$$

Hence $u \in L^q\left(\mathbb{R}^N\right)$ for all $q > 0$ large.

Obviously u satisfies the linear problem

$$-\triangle u + u = F(x) = f(u) + \mu h(x), \qquad x \in H^1\left(\mathbb{R}^N\right).$$

Choose $q > \max\left\{\frac{Np}{2}, \frac{2N}{N-2}\right\}$. By the Hölder's inequality in $B_2(x)$ we get

$$\|u\|_{L^2(B_2(x))} \le C\,\|u\|_{L^q(B_2(x))}. \tag{4.4}$$

The assumption Remark 1.9 and f_2) yield

$$\begin{aligned} \|F(x)\|_{L^{\frac{q}{p}}(B_2(x))} &= \left(\int_{B_2(x)} [f(u)+\mu h(x)]^{\frac{q}{p}}\, dx\right)^{\frac{p}{q}} \\ &\le \left(\int_{B_2(x)} (C_1 u^p + f'(0)u + \mu^* h(x))^{\frac{q}{p}}\, dx\right)^{\frac{p}{q}} \\ &\le \left(C\int_{B_2(x)} C_1 u^q + f'(0)u^{\frac{q}{p}} + (\mu^* h(x))^{\frac{q}{p}}\, dx\right)^{\frac{p}{q}} \\ &\le C \qquad \text{for all } q \text{ large enough.} \end{aligned} \tag{4.5}$$

It is deduced by elliptic regularity theory that $u \in C^{2,\alpha}(\mathbb{R}^N)$. By [GT, Theorem 8.24] we have

$$\|u\|_{C^{\alpha}(B_1(x))} \le C;$$

then $u(x) \to 0$ as $|x| \to \infty$ since $u \in L^q(\mathbb{R}^N)$. By [GT, Theorem 8.32],

$$\|u\|_{C^{1,\alpha}(B_1(x))} \le C\left(\|u\|_{C^{\alpha}(B_2(x))} + \|h(x)\|_{0\,B_2(x)}\right) \tag{4.6}$$

(4.5)–(4.6) give $|(\nabla u)(x)| \to 0$ as $|x| \to \infty$, $u(x) \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$.

Part (ii) can be established as in [S, Proposition 4.4]. □

Proposition 4.2. *Suppose I) − III) and h). Let* $h(x) \in C^{\alpha}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. *Then* $U \subset L^{\infty}(\mathbb{R}^N)$ *and* U *is uniformly bounded in* $L^{\infty}(\mathbb{R}^N)$, *where* U *is given by (1.3).*

Proof. By elliptic regularity theory [GT] we can deduce that $U \subset C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$. Suppose on the contrary that there is a sequence $\{u_n\} \subset U$ such that $\sup_{x\in\mathbb{R}^N} u_n \longrightarrow +\infty$. Take

$$M_n = \sup_{x\in\mathbb{R}^N} u_n(x) = u_n(x_n), \quad y = \alpha x + x_n, \quad w_n(x) = \frac{1}{M_n} u_n(\alpha x + x_n). \tag{4.7}$$

Where α is some constant to be determined later. Clearly, $0 \le w_n(x) \le 1$ and $w_n(0) = 1$. Because u_n are the solutions of $(1.2)_{\mu}$, we have

$$-M_n \frac{1}{\alpha^2} \triangle w_n(x) + M_n w_n(x) = f(M_n w_n(x)) + \mu_n h(\alpha x + x_n). \tag{4.8}$$

Letting $\alpha = M_n^{\frac{p-2}{2}}$ we have

$$-\triangle w_n(x) + M_n^{2-p} w_n(x) = M_n^{1-p} f(M_n w_n(x)) + \frac{\mu_n}{M_n^{p-1}} h(\alpha x + x_n). \tag{4.9}$$

From $0 \leq w_n(x) \leq 1$ and the elliptic regularity theory we deduce that $w_n(x)$ is bounded in $C^{2,\alpha}(\mathbb{R}^N)$. So we can suppose that

$$w_n(x) \longrightarrow w_\infty \ in \ C^2(\mathbb{R}^N) \ as \ n \longrightarrow \infty,$$

and hence w_∞ is a nontrivial positive solution of

$$-\triangle w = B w^{p-1} \ with \ \lim_{|x| \longrightarrow \infty} w(x) = 0 \ and \ w(0) = 1,$$

which is impossible by [CG], [CL], [GS]. □

Proposition 4.3. *Let $u \in C^{2,\alpha}\left(\mathbb{R}^N\right)$ be a solution of*

$$\begin{cases} -\triangle u + u = f(u) + \mu h(x) & in \ \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right), & u > 0 \ in \ \mathbb{R}^N, \end{cases} \tag{4.10}$$

and let $v \in H^1\left(\mathbb{R}^N\right)$ be a supersolution of (4.10). *Recall*

$$\lambda_1(u) = \inf\left\{ \int_{\mathbb{R}^N} |\nabla v|^2 + (1 - f'(0))v^2 dx \,\middle|\, v \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} (f'(u) - f'(0))v^2 \, dx = 1 \right\}. \tag{4.11}$$

Then:

(i) $v \geq u$ *if* $\lambda_1(u) > 1$; $v = u$ *if* $\lambda_1(u) = 1$.

(ii) *If* $\lambda_1(u) < 1$, $v(x) \geq u(x)$, $v(x) \not\equiv u(x)$ *does not hold for all* $x \in \mathbb{R}^N$.

Proof. For the case $\lambda_1(u) > 1$, if the conclusion were not true, we would have the set $G = \left\{x \in \mathbb{R}^N, \ v(x) < u(x)\right\} \neq \varnothing$ and $meas(G) > 0$. It is clear from (4.10) and the fact that v is a supersolution of (4.10) that

$$-\triangle u + (1 - f'(0))u = f(u) - f'(0)u + \mu h(x), \tag{4.12}$$

$$-\triangle v + (1 - f'(0))v \geq f(v) - f'(0)v + \mu h(x).$$

From the convexity of $f(u)$ with respect to u we have

$$
\begin{aligned}
-\triangle(v-u)+(1-f'(0))(v-u) &\geq f(v)-f(u)-f'(0)(v-u) \qquad (4.13)\\
&=[f'(u)-f'(0)](v-u)+f''(u+\Theta(v-u))(v-u)^2\\
&\geq [f'(u)-f'(0)](v-u)
\end{aligned}
$$

for some $\Theta(x)\in[0,1]$. Set

$$
w(x)=\begin{cases}-(v-u)(x), & x\in G,\\ 0 & \text{otherwise.}\end{cases}
$$

Hence

$$
-\triangle w+(1-f'(0))w\leq [f'(u)-f'(0)]w,
$$

which gives

$$
\int_{\mathbb{R}^N}\left(|\nabla w|^2+(1-f'(0))w^2\right)dx\leq \int_{\mathbb{R}^N}[f'(u)-f'(0)]w^2\,dx.
$$

Thus

$$
1\geq \int_{\mathbb{R}^N}\left(|\nabla w|^2+(1-f'(0))w^2\right)dx\Big/\int_{\mathbb{R}^N}[f'(u)-f'(0)]w^2\,dx\geq \lambda_1(u)>1,
$$

a contradiction.

It is known by Proposition 4.1 that $u(x)\to 0$ as $|x|\to\infty$, so $\lambda_1(u)$ is attained by some $\varphi_1>0$ in view of Lemma 2.1. Relation (4.13) leads to

$$
\begin{aligned}
&\lambda_1(u)\int_{\mathbb{R}^N}(v-u)[f'(u)-f'(0)]\varphi_1\,dx\\
&\quad=\int_{\mathbb{R}^N}-\triangle(v-u)\varphi_1+(1-f'(0))(v-u)\varphi_1\,dx\\
&\quad\geq \int_{\mathbb{R}^N}[f'(u)-f'(0)](v-u)\varphi_1\,dx+\int_{\mathbb{R}^N}f''(u+\Theta(v-u))(v-u)^2\varphi_1\,dx
\end{aligned}
$$

for some $\Theta(x)\in[0,1]$. So

$$
(\lambda_1(u)-1)\int_{\mathbb{R}^N}(v-u)[f'(u)-f'(0)]\varphi_1\,dx\geq \int_{\mathbb{R}^N}f''(u+\Theta(v-u))\left(v-u^2\right)\varphi_1\,dx\geq 0. \qquad (4.14)
$$

If $\lambda_1(u)=1$, (4.14) yields $u=v$; if $\lambda_1(u)<1$, $v\geq u$, then we infer from (4.14) that $v=u$, and the conclusion follows. □

Now we may prove

Proposition 4.4. *Let u be a positive solution of $(1.2)_\mu$. Then u is the minimal positive solution of $(1.2)_\mu$ if and only if $\lambda_1(u) \geq 1$, and the equality holds if and only if $u = u_{\mu^*}$ where u_{μ^*} is the unique positive solution of $(1.2)_{\mu^*}$.*

Proof. It follows from Proposition 4.3 that the nonnegative solution u such that $\lambda_1(u) > 1$ is unique, so if $\lambda_1(u) > 1$, from Lemma 2.1 u must be the minimal solution u_μ of $\mu \in (0, \mu^*)$. If $\lambda_1(u) = 1$, from Lemma 2.1 we deduce $u = u_{\mu^*}$, the unique solution of $(1.2)_{\mu^*}$. □

Proposition 4.5. *Suppose $I) - II)$, U_μ and u_μ are the solutions of $(1.1)_\mu$, then for any $\delta > 0$, there exist $C, R > 0$ such that*

$$U_\mu(x) - u_\mu(x) \leq C \exp\left(-\left(1 - f'(0) - \delta\right)|x|\right) \qquad \textit{for } |x| \geq R, \tag{4.15}$$

Proof. From Proposition 4.1 we have

$$\lim_{|x| \to \infty} (U_\mu(x) - u_\mu(x)) = 0.$$

Let $\beta = (1 - f'(0) - \delta)^{\frac{1}{2}}$ for some $\delta > 0$ so that $1 - f'(0) - \delta > 0$. Set $w(x) = U_\mu(x) - u_\mu(x)$, then w is the positive solution of

$$-\triangle w + w = f(w + u_\mu) - f(u_\mu), \quad w \in H^1(\mathbb{R}^N). \tag{4.16}$$

Since $w(x) \to 0$ as $|x| \to \infty$, by $I)$, $II)$ there exists an $R > 0$ such that

$$1 - f_t'(0) - \frac{f(w + u_\mu) - f(u_\mu) - f'(0) w}{w} \geq 1 - f'(0) - \frac{\delta}{2} \tag{4.17}$$

for $|x| \geq R$. Let $v(x) = m \exp\left(-\beta\left(|x| - R\right)\right)$, where $m = \max\{w(x) \mid \{x\} = R\} > 0$. For any $M > R$, set

$$\Omega(M) = \{x \in \Omega \mid R < |x| < M \text{ and } w(x) > v(x)\}.$$

Then $\Omega(M)$ is open. For any $x \in \Omega(M)$, we have

$$\begin{aligned}\triangle(v-w)(x) &= \left(\beta^2 - \beta(N-1)|x|^{-1}\right)v(x) \\ &\quad - \left[1 - f'(0) - \frac{f(w+u_\mu) - f'(0)w - f(u_\mu)}{w}\right]w \\ &\le \beta^2 v(x) - \left[1 - f'(0) - \frac{\delta}{2}\right]w(x) \\ &= (1 - f'(0) - \delta)v(x) - \left(1 - f'(0) - \frac{\delta}{2}\right)w(x) \\ &= (1 - f'(0) - \delta)(v(x) - w(x)) - \frac{\delta}{2}w(x) \\ &\le (1 - f'(0) - \delta)(v(x) - w(x)) < 0.\end{aligned}$$

By the maximum principle, we obtain, for $x \in \Omega(M)$,

$$\begin{aligned}v(x) - w(x) &\ge \min\{(v-w)(x) \mid x \in \partial\Omega(M)\} \\ &= \min\{0, \min\{(v-w)(x) \mid |x| = M\}\}.\end{aligned}$$

Since $\lim_{|x|\to\infty} w(x) = \lim_{|x|\to\infty} v(x) = 0$, this yields, by letting $M \to +\infty$, that

$$v(x) \ge w(x) \qquad \text{for } |x| \ge R,$$

hence (4.15) follows. □

Lemma 4.6. *Let $h(x) \in C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. Then for any $g(x) \in C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ problem $(3.13)_\mu$ has a solution $w \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ for all $\mu \in (0, \mu^*)$ (again we suppose here $u_0 \equiv 0$).*

Proof. From Lemma 2.2 we know that $(3.13)_\mu$ has a solution $w \in H^1(\mathbb{R}^N)$. By the assumptions on h and g, it is known from [[S]; Proposition 4.3] that $w \in H^2(\mathbb{R}^N)$. The standard elliptic regularity theory yields $w \in C^{2,\alpha}(\mathbb{R}^N)$. □

Similarly, we can prove that

Lemma 4.7. *Let $h(x) \in C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, then for any $g(x) \in C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ problem (3.1) has a solution $w \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$.*

Proof of Theorem 1.4. The conclusion i) comes immediately from Proposition 4.2. As for ii), we define

$$G : \mathbb{R}^1 \times C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \longrightarrow C^{\alpha}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \tag{4.18}$$

by

$$G(\mu,\ u) = \triangle u - u + f(\ u^+) + \mu h(x), \tag{4.19}$$

where $C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ and $C^{\alpha}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ are endowed with the natural norms. Then they become a Banach spaces. It can be verified that $F(\mu,\ u)$ is differentiable. From Lemma 4.6 and Lemma 4.7 we know that for $\mu \in (0, \mu^*)$,

$$G_u(\mu,\ u_\mu)w = \triangle w - w + f'_u(\ u_\mu)w$$

is an isomorphism of $C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ onto $C^{\alpha}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. It follows from Implicit Function Theorem that the solutions of $G(\mu,\ u) = 0$ near $(\mu,\ u_\mu)$ are given by a continuous curve.

Now we are going to prove that $(\mu^*,\ u_{\mu^*})$ is a bifurcation point in $C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ by using an idea in [KLO]. To this end, we need the following bifurcation theorem [CR]:

Theorem F. Let X, Y be Banach spaces. Let $(\bar{\lambda}, \bar{x}) \in \mathbb{R} \times X$ and let G be a continuously differentiable mapping of an open neighborhood of $(\bar{\lambda},\ \bar{x})$ into Y. Let the null-space $N(G_x(\bar{\lambda},\ \bar{x})) = span\{x_0\}$ be one-dimensional and $codim R(G_x(\bar{\lambda},\ \bar{x})) = 1$. Let $G_\lambda(\bar{\lambda},\ \bar{x}) \notin R(G_x(\bar{\lambda},\ \bar{x}))$. If Z is the complement of $span\{x_0\}$ in X, then the solutions of $G(\lambda,\ x) = G(\bar{\lambda},\ \bar{x})$ near $(\bar{\lambda},\ \bar{x})$ form a curve $(\lambda(s),\ x(s)) = (\bar{\lambda} + \tau(s),\ \bar{x} + sx_0 + z(s))$, where $s \longrightarrow (\tau(s),\ z(s)) \in \mathbb{R} \times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = z(0) = z'(0) = 0$.

we define G as (4.18), (4.19). We show that at the critical point $(\mu^*,\ u_{\mu^*})$, the Theorem F applies. Indeed, from Lemma 2.3, problem (2.1) has a solution $\phi_1 > 0$ in $\mathbb{R}^N$. $\phi_1 \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ if $h \in C^{\alpha}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. Thus $G(\mu^*,\ u_{\mu^*})\phi = 0$, $\phi \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ has a solution $\phi_1 > 0$. This implies that $N(G_u(\mu^*,\ u_{\mu^*})) = span\{\phi_1\} = 1$ is one dimensional and $codim R(G_u(\mu^*,\ u_{\mu^*})) = 1$ by the Fredholm alternative. It remains to check that $G_\mu(\mu^*,\ u_{\mu^*}) \notin R(G_u(\mu^*,\ u_{\mu^*}))$.

By contropositive, it would imply the existence of $v(x) \not\equiv 0$ such that

$$\triangle v - v + f'_u(\ u_{\mu^*})v = -h(x), \quad v \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$$

. From $G_u(\mu^*,\ u_{\mu^*})\phi_1 = 0$ we conclude that $\int_{\mathbb{R}^N} h(x)\phi_1 dx = 0$. This is impossible because $h(x) \geq 0$, $h(x) \not\equiv 0$ and $\phi_1(x) > 0$ in $\mathbb{R}^N$.

Applying Theorem F we conclude that $(\mu^*,\ u_{\mu^*})$ is the bifurcation point near which, the solutions of $(1.2)_\mu$ form a curve $(\mu^* + \tau(s),\ u_{\mu^*} + s\phi_1 + z(s))$ with s near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$. We claim that $\tau''(0) < 0$ which implies that the bifurcation curve turns strictly to the left in $(\mu,\ u)$ plane. Since $\mu = \mu^* + \tau(s)$, $u = u_{\mu^*} + s\phi_1 + z(s)$ in

$$-\triangle u + u - f(\ u) - \mu h(x) = 0,\ u > 0, \quad u \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N). \tag{4.20}$$

Differentiate (4.20) in s twice we have

$$-\triangle u_{ss} + u_{ss} - f''(\ u)u_s^2 - f'(\ u)u_{ss} - \mu_{ss}h(x) = 0.$$

Set here $s = 0$ and use that $\tau'(0) = 0$, $u_s = \phi_1(x)$ and $u = u_{\mu^*}$ as $s = 0$ we obtain

$$-\triangle u_{ss} + u_{ss} - f''_u(\ u_{\mu^*})\phi_1^2 - f'_u(\ u_{\mu^*})u_{ss} + \tau''(0)h(x) = 0. \tag{4.21}$$

Multiplying

$$G_u(\mu^*,\ u_{\mu^*})\phi_1 = 0$$

by u_{ss}, and (4.21) by ϕ_1, integrating and subtracting the results we obtain

$$\int_{\mathbb{R}^N} f''(\ u_{\mu^*})\phi_1^3 dx + \tau''(0)\int_{\mathbb{R}^N} h(x)dx = 0$$

which immediately gives $\tau''(0) < 0$ because $f''_u(\ u) \geq 0$ for all $u \geq 0$.

Thus

$$u_\mu \longrightarrow u_{\mu^*} \ \ in \ \ C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \ as \ \mu \longrightarrow \mu^*,$$

$$U_\mu \longrightarrow u_{\mu^*} \ \ in \ \ C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \ as \ \mu \longrightarrow \mu^*.$$

Using Lemma 4.6 - Lemma 4.7 and Proposition 4.2, the implicit function theorem and the uniqueness of positive ground state solution of $(1.2)_0$ we can easily prove that

$$u_\mu \longrightarrow 0 \ in \ C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \ as \ \mu \longrightarrow 0,$$

and

$$U_\mu \longrightarrow U_0 \ in \ C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \ as \ \mu \longrightarrow 0.$$

□

5 A Uniqueness Result

In this section we shall always assume that $h(x)$ satisfies the conditions of Theorem 1.3.

We first give a Pohozaev identity. Let

$$g(u_\mu, u) = f(u_\mu + u) - f(u_\mu) \tag{5.1}$$

$$G(u_\mu, u) = \int_0^u g(u_\mu, s)\, ds \tag{5.2}$$

The following Lemma can be found in [DL1]

Lemma 5.1. *If $u \in H^1(\mathbb{R}^N)$ is a positive solution of*

$$-\triangle u + u = g(u_\mu, u) \tag{5.3}$$

then

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx = \frac{2N}{N-2} \int_{\mathbb{R}^N} G(u_\mu, u) - \frac{1}{2} u^2 dx + \frac{2}{N-2} \int_{\mathbb{R}^N} \frac{\partial G}{\partial u_\mu} (\nabla u_\mu \cdot x) dx. \tag{5.4}$$

Lemma 5.2. *Let $\frac{4}{N-2} f'(t) - u f''(t) \le 0$ and $f''(t)$ is nonincreasing. Then (5.3) has no positive solutions if $h(x)$ satisfies (1.4) and μ is small enough.*

Proof. If (5.3) possesses a positive solution u, by Lemma 5.1, u satisfies (5.4). Using (5.3) we deduce

$$\int_{\mathbb{R}^N} (u g(u_\mu, u) - u^2) dx = \frac{2N}{N-2} \int_{\mathbb{R}^N} (G(u_\mu, u) - \frac{1}{2} u^2) dx + \frac{2}{N-2} \int_{\mathbb{R}^N} \frac{\partial G}{\partial u_\mu} (\nabla u_\mu \cdot x) dx$$

and hence

$$0 = \int_{\mathbb{R}^N} [\frac{2N}{N-2} G(u_\mu, u) - u g(u_\mu, u) - (\frac{N}{N-2} - 1) u^2] dx + \int_{\mathbb{R}^N} \frac{\partial G}{\partial u_\mu} (\nabla u \cdot x) dx. \tag{5.5}$$

Denote

$$H(u_\mu, u) = \frac{2N}{N-2}G(u_\mu, u) - ug(u_\mu, u) - (\frac{N}{N-2} - 1)u^2,$$

then

$$H'_u(u_\mu, u) = \frac{N+2}{N-2}g(u_\mu, u) - ug'_u(u_\mu, u) + 2(1 - \frac{N}{N-2})u$$

and

$$\begin{aligned} H''_{uu}(u_\mu, u) &= \tfrac{4}{N-2}g'_u(u_\mu, u) - ug''_{uu}(u_\mu, u) + 2(1 - \tfrac{N}{N-2}) \\ &= \tfrac{4}{N-2}f'(u_\mu + u) - uf''(u_\mu + u) + 2(1 - \tfrac{N}{N-2}) \\ &= \tfrac{4}{N-2}f'(u_\mu + u) - (u + u_\mu)f''(u_\mu + u) \\ &\quad + u_\mu(f''(u_\mu + u) - f''(u_\mu)) + u_\mu f''(u_\mu) + 2(1 - \tfrac{N}{N-2}) \\ &\le u_\mu f''(u_\mu) + 2(1 - \tfrac{N}{N-2}). \end{aligned}$$

By the assumption II) and the fact that $\lim_{\mu\to 0} u_\mu = 0$ we deduce that $H''_{uu}(u_\mu, u) \le 0$ and hence $H(u_\mu, u) \le 0$ for all $u > 0$ because $H(u_\mu, 0) = 0$ and $H'_u(u_\mu, 0) = 0$. Now (5.5) becomes

$$\int_{\mathbb{R}^N} \frac{\partial G}{\partial u_\mu}(\nabla u_\mu \cdot x)dx > 0. \tag{5.6}$$

On the other hand, by (1.4) we know from [GNN, Li, LN] that

$$(\nabla u_\mu \cdot x) < 0.$$

It is easy to verify that $\frac{\partial G}{\partial u_\mu} \ge 0$ for all $u_\mu \ge 0$, $u \ge 0$. Hence

$$\int_{\mathbb{R}^N} \frac{\partial G}{\partial u_\mu}(\nabla u_\mu \cdot x)dx \le 0$$

This is contradictory to (5.6). □

Corollary 5.3. *Let $N \ge 6$, $p = \frac{2N}{N-2}$, Then (5.3) has no solutions in $H^1(\mathbb{R}^N)$ if (1.4) holds and μ is small enough.*

The Proof of Theorem 1.5: It follows from Theorem 1.1 and Theorem 1.4 that $(1.1)_\mu$ possesses a minimal solution u_μ if $\mu \in (0, \mu_{**})$ for some positive constant μ_{**}. If $(1.1)_\mu$ has another solution U_μ and $U_\mu \not\equiv u_\mu$, then $U_\mu \ge u_\mu$ and $v_\mu \equiv U_\mu - u_\mu \ge 0$ must be a solution of (5.3). Strong maximum principle implies that v_μ is a positive solution of (5.3). This is contradictory to Lemma 5.2.

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