TRAVELLING WAVE SOLUTIONS OF THE ONE DIMENSIONAL DIFFUSION-REACTION EQUATION

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Abstract. We study the existence, uniqueness, asymptotic expansion and stability of the travelling wave solution of the one dimensional Diffusion- reaction equation under some degenerate conditions.

Part 1. Introduction

In this paper, we study the existence, uniqueness, asymptotic behaviors and the stability properties of the travelling wave solutions of the one-dimensional reaction-diffusion equation

$$u_t = u_{xx} + f(u), \qquad -\infty < x < \infty$$

Here $f(0) = f(1) = 0$ and it was pointed out in [1] the existence, uniqueness and asymptotic behaviors of the travelling wave solutions depend very critically on the behaviors of f near 0 and 1 . The assumption $(f'(0))^2 + (f'(1))^2 \neq 0$ were made in the past and five different kinds of nonlinearites were considered, see [13] for details. Now if we slightly relax the requirement from $(f'(0))^2 + (f'(1))^2 \neq 0$ to $f'(0) = f'(1) = 0$, things are a little bit subtle, indeed, we do not have exponential solutions at both ends (negative infinity and infinity). Even in the case $f'(0) = 0$, $f'(1) < 0$ and if c is bigger than some critical value c^*, the travelling wave solutions do not decay exponentially. See [4]. The motivative example of this paper is the so called Josephson equation:

$$\phi'' + \sigma\phi' + \sin\phi = 1$$

It was proved in [5] that this kind of equation admits connection between the equilibrium $(-\frac{\pi}{2},\, 0)$ and $(\frac{3\pi}{2},\, 0)$ $(mod\, 2\pi)$ for $\sigma \geq \sigma_0 > 0$ in the phase plane, in other words, there are travelling wave solutions from $-\infty$ to ∞. In this paper, we are considering the more general nonlinearity f with $f'(0) = f'(1) = 0$. Travelling wave solutions are solutions of the form $u = u\,(x + ct)$ satisfying $u'' - cu' + f(u) = 0$. In one dimensional case, it is an O.D.E. system with boundary condition $u(-\infty) = 0$, $u(+\infty) = 1$. Therefore, Josephsen equation falls into our category.

We are able to prove the existence uniqueness and stability properties of the travelling wave solutions to (???). Being lack of linear term at infinity for f, the linearized equation will help us less. By making use of the sliding method developed in [1] [2], and construction of super and sub-solution, we can prove the existence of travelling wave solutions and comparison theorems and therefore the asymptotic expansion and the uniqueness of the travelling wave solution is also derived.

A spectrum analysis reveals that the "linearized" operator

$$Lu = u'' - cu' + f'(\phi)u$$

has continuous spectrum touching the origin from the left in complex plane C if we consider operator L in L_∞ space, the reason that the operator L has continuous spectrum is that we are considerring our operator in the unbounded domain R. Hence in order to obtain our stability results we must consider operator L in some weighted Banach space which is relatively small. The idea is simple, if we consider weighted Banach space, we can get nicer spectrum properties; on the one hand, we can push the continuous spectrum all the way to the left, on the other hand, we have only discrete eigenvalues left in the remaining region of complex plane. To prove that operator L in some weighted Banach space has eigenvalues with nonpositive real parts, the following assumptions on $f'(0)$ and $f'(1)$ were made in [6] [7] [12], to cite a few,

1. $(f'(0))^2 + (f'(1))^2 \neq 0$, or,
2. the travelling wave solutions decay exponentially.

KPP and Fitzhugh-Nagumo equation belong to type 1 and Burges equation satisfies condition 2. Therefore, a delicate investigation is required if neither condition 1 nor 2 holds. We are able to locate our spectrum of L in some weighted Banach space, C_{σ_1,σ_2} (see the definiton in section 4), actually, we are able to prove that the linearized operator L has spectrum with negative real parts in space C_{σ_1,σ_2}. The idea behind this is that we can first fix our continuous spectrum in space C_{σ_1,σ_2} and then we consider operator L in a relatively bigger space $C_{0,\frac{c}{2}}$ and we also can obtain the location of the eigenvalues of L in $C_{0,\frac{c}{2}}$ by the containning relation of the point spectrum of our operator L in space C_{σ_1,σ_2} and space $C_{0,\frac{c}{2}}$. Once this is done, the stability results will follow. Our derivation of the spectrum is different from those in [6], [7], [12]. Actually we use two different weighted Banach spaces to get our results which have not been seen in literatures.

Now we state our main results.

Theorem 0.1. *Suppose f satisfies conditions i)-v), then there exists $c* > 0$ and a function $\phi(\xi)$, $\xi \in R$, such that*

$$\begin{cases} \phi'' - c^*\phi' + & f(\phi) = 0 \\ \phi(-\infty) = 0, & \phi(+\infty) = 1 \end{cases}$$

and $\phi'(\xi) > 0$ on R. The constant c^* is uniquely determined by (*).

We then prove the following

Theorem 0.2. *There is no solution of (???) for any $0 < c < c^*$.*

and

Theorem 0.3. *Suppose the conditions of Theorem 1 are satisfied, then for every $c > c^*$, c^* is given by Theorem 1, then*

$$\begin{cases} u'' - cu' + & f(u) = 0 \\ u(-\infty) = 0, & u(+\infty) = 1 \end{cases}$$

has a solution.

About the Asymptotic behavours of the travelling wave solutions, we have

Theorem 0.4. *The travelling wave solution of system (???) has the following asymptoti behavoirs*

1. $c = c^*$.

$$\begin{array}{ll} \phi(\xi) = H(\frac{1}{c^*}\xi) + \circ(1) & as\, \xi \to +\infty; \\ \phi(\xi) = Ae^{c^*\xi} + \circ(e^{c^*\xi}) & as\, \xi \to -\infty. \end{array}$$

2. $c > c^*$,

We have the same expansion as the first expansion but with c^* *replaced by* c.

Finally the stability Theorem,

Theorem 0.5. *The travelling wave solution* $\phi(\xi)$ *is asymptotically stable in space* C_{σ_1,σ_2}.

This paper is organized as follows: In section 2, we prove the existence, uniqueness; in section 3, we give the asymptotic behaviors and in section 4, we prove the stability of our travelling wave solutions.

Part 2. Existence and Uniqueness

The problems under investigation are the existence, uniqueness, asymptotic behaviors of the travelling wave solutions of the one-dimentinal reaction-diffusion equation:

$$u_t = u_{xx} + f(u) \tag{0.1}$$

Where f satisfies

(i) $f(0) = f(1) = 0$;

(ii) $f'(0) = f'(1) = 0$;

(iii) f is Lipschitz continuous on $[0,1]$ and is of $c^{1,\alpha}$ for u near 0 and 1;

(iv) $f'(s) > 0$, $f'(1-s) < 0$ for $s > 0$ small enough;

(v) $f > 0$ on $(0,1)$.

More specific conditions will be given on f later on, if necessary.

We say function $\phi(\xi) = \phi(x+ct)$ is a travelling wave solution of (???) if there exists a constant $c > 0$, s.t

$$\begin{cases} \phi'' - c\phi' + & f(\phi) = 0 \\ \phi(-\infty) = 0, & \phi(\infty) = 1 \end{cases} \tag{0.2}$$

1. Existence.

Now we prove the existence of travelling wave solutions of (???) under conditions (i)-(v). To do so, we need the following results:

Let

$$\xi_\theta = \begin{cases} 0, & \xi \in [0,\ \theta); \\ smooth, & \xi \in (0,\ \frac{1}{2}); \\ 1, & \xi \in [\frac{1}{2},\ 1]. \end{cases}$$

and $f_\theta = f \cdot \xi_\theta$ be the cut-off function of f.

Consider the following auxiliary problem

$$\begin{cases} \ddot{u}_\theta - c_\theta \dot{u}_\theta + f_\theta = 0 \\ u_\theta(-\infty) = 0, & u_\theta(+\infty) = 1 \end{cases} \tag{0.3}$$

We have

Lemma 0.6. *There exists a solution* $v_\theta : R \rightarrow [0,\ 1]$ *and* $c_\theta > 0$ *of problem (???).* v_θ *is of class* C^1 *and of class* C^2 *on* $R \setminus \{\xi_0\}$, ϕ_θ *is of monotone increasing on* R. *Furthermore,* v_θ *and* c_θ *are uniquely determined from (0.2) (up to a translation of the origing). There are positive constants* A *and* δ, *such that*

$$0 < v_\theta(\xi) \leq Ae^{\delta\xi}, \forall \xi < 0$$

and if $f'(1) = 0$ *and* $f(u) \leq C(1-u)^p$, $p > 1$, *then,*

$$v_\theta(\xi) = H(\frac{1}{c}\xi) + \circ(1)$$

for $\xi_0 > 0$ *sufficiently large, and* $\xi > \xi_0$ *where* $H = F^{-1}$, $F = \int_{u_0}^{u} \frac{ds}{f(s)}$ *for* u *close to* 1 *or equivalently,* $\xi \rightarrow +\infty$.

Proof. The first part of the lemma is Thorem 3.1 of [3].

Now we prove the second part, by the first part, for some (c_θ, v_θ). We have

$$\begin{cases} v_\theta'' - c_\theta v_\theta' + f(v_\theta) = 0 \\ v_\theta(-\infty) = 0, v_\theta(+\infty) = 1 \end{cases}$$

and $v_\theta'(\xi) > 0$ for $\xi \in R$.

We have

$$\frac{v_\theta''}{v_\theta'} - c_\theta + \frac{f(v_\theta)}{v_\theta'} = 0$$

then

$$\frac{f(v_\theta)}{v_\theta'} = \frac{1}{c_\theta - \frac{v_\theta''}{v_\theta'}}$$

By lemma???(please see section 3 for details)

$$\frac{v_\theta''}{v_\theta'} \rightarrow 0$$

as $\xi \rightarrow +\infty$.

Then, by Taylor expansion,

$$\begin{aligned} \frac{1}{c_\theta - \frac{v_\theta''}{v_\theta'}} &= \frac{1}{c_\theta(1 - \frac{1}{c_\theta}\frac{v_\theta''}{v_\theta'}} \\ &= \frac{1}{c_\theta}\frac{1}{1 - \frac{1}{c_\theta}\frac{v_\theta''}{v_\theta'}} \\ &= \frac{1}{c_\theta}(1 + \frac{1}{c_\theta}\frac{v_\theta''}{v_\theta'} + (\frac{1}{c_\theta}\frac{v_\theta''}{v_\theta'})^2 + \cdots) \end{aligned}$$

then,

$$\begin{aligned} \frac{v_\theta'}{f(v_\theta)} &= \frac{1}{c_\theta}[1 + \frac{1}{c_\theta}\frac{v_\theta''}{v_\theta'} + (\frac{1}{c_\theta}\frac{v_\theta''}{v_\theta'})^2 + \cdots] \\ \int_{\xi_0}^{\xi} \frac{v_\theta'}{f(v_\theta)} ds &= \int_{\xi_0}^{\xi} \frac{1}{c_\theta}[1 + \frac{1}{c_\theta}(\frac{v_\theta''}{v_\theta'}) + (\frac{1}{c_\theta}\frac{v_\theta''}{v_\theta'})^2 + \cdots] ds \\ \int_{u(\xi_0)}^{u(\xi)} \frac{ds}{f(s)} &= \int_{\xi_0}^{\xi} \frac{1}{c_\theta}[1 + \frac{1}{c_\theta}(\frac{v_\theta''}{v_\theta'}) + (\frac{1}{c_\theta}\frac{v_\theta''}{v_\theta'})^2 + \cdots] ds \end{aligned}$$

Denote $F(u) = \int_{u(\xi_\theta)}^{u(\xi)} \frac{ds}{f(s)}$, then by $F'(u) = \frac{1}{f(u)} > 0$, we see that F is invertible. Let $F^{-1} = H$, we have

$$\begin{aligned} u(\xi) &= H(\int_{\xi_0}^{\xi} \tfrac{1}{c_\theta}(1+\tfrac{1}{c_\theta}(\tfrac{v_\theta''}{v_\theta'})+(\tfrac{1}{c_\theta}\tfrac{v_\theta''}{v_\theta'})^2+\cdots]ds \\ &= H(\tfrac{1}{c_\theta}(\xi-\xi_0)[1+\circ(1)]) \end{aligned}$$

because $f(u) \to 0$ as $u \to 1$, we then have

$$u(\xi) = H(\frac{1}{c_\theta}(\xi-\xi_0)[1+\circ(1)]))$$

then, after a shift of the origin,

$$u(\xi) = H(\frac{1}{c_\theta}(\xi)) + \circ(1)$$

Next, we need to compare our solution $v_\theta(\xi)$ of (0.2) with $e^{-c\xi}$, where $c>0$ is any constant. □

Lemma 0.7. *For any constant $c>0$, we have*

$$v_\theta(\xi) \le 1 - Ae^{-c\xi} \quad for\, \xi > 0\, and\, sufficiently\, large$$

here $A>0$ is some constant.

Proof. By condition ii),

$$\begin{aligned} f(s) &\le B(1-s)^p \\ &\le D(1-s) \end{aligned}$$

for some $s>0$ small enough and some constant $D>0$. We have

$$\begin{aligned} F(u) &= \int_{u(\xi_0)}^{u(\xi)} \tfrac{ds}{f(s)} \\ &\ge \int_{u(\xi_0)}^{u(\xi)} \tfrac{ds}{D(1-s)} \\ &\doteq F_1(u) \end{aligned}$$

Hence, $F^{-1}(u) \le F_1^{-1}(u)$, that is,

$$v_\theta(\xi) \le -Ae^{-c\xi}$$

for any constant $c>0$. □

Now we prove that c_θ in (0.2) has a upper bound as $\theta \to 0$. To do so, the following lemmas are needed.

Lemma 0.8. *Assume that functions v and z satisfy*

$$\begin{aligned} v''-cv'+f(v) &\ge 0 \\ z''-cz'+f(z) &\le 0 \end{aligned} \qquad on\ [-a,a]$$

and

$$\begin{aligned} v(-a) &< z(\xi), \qquad \xi \in (-a,a]; \\ v(\xi) &< z(a), \qquad \xi \in [-a,a) \end{aligned}$$

then

$$v(\xi) \le z(\xi), \qquad \xi \in (-a,a).$$

Furthermore, if

$$z(a) > v(a), \quad z(-a) > v(-a),$$

then

$$v(\xi) < z(\xi), \qquad \xi \in [-a, a].$$

Proof. The first part of this Lemma was proved in [1] for higher dimensional case, here we adapt their proof to one dimensional case.

Let $S = (-a,\, a)$ be an interval, we are going to use the sliding method. Shift z to the left, i.e, for $0 \le r \le a$, consider

$$z^r(\xi) = z(\xi + r) \qquad in\ S^r = (-a-r,\, a-r)$$

We have

$$\begin{aligned} &z^r(a-r) = z(-a-r+r) = z(a),\\ &z^r(-a-r) = z(-a) \end{aligned}$$

and if $r = 0$, $S^r = s^0$; if $r = 2a$, $\overline{S^r} \bigcap \overline{S} = \{-a\}$, $z^r(a) = z(-a) > v(-a)$

Now decreasing r, for any r in $0 < r < 2a$, we have $z^r = .v$ on $\partial(\overline{S^r} \bigcap \overline{S}) = \{\xi = -a\} \bigcup \{\xi = a - r\}$. Keep decreasing r, suppose we can find $\bar{r} \in (0, 2a)$ such that z^r touches v, then $z^r \ge v$ in $\overline{S^r} \bigcap \overline{S}$ and we have $z^r = v$ some point in $\overline{S^r} \bigcap \overline{S}$, see figures below,

Hence in $\overline{S^r} \bigcap \overline{S}$,

$$\begin{aligned} &w_r = z^r - v \ge 0\\ &\ddot{w}_r - c\dot{w}_r + f(z^r) - f(v) \le 0 \end{aligned}$$

i.e.

$$\ddot{w}_r - c\dot{w}_r + d(\xi) w_r \le 0$$

where $d(\xi) = f'(v(\xi))$.

Let $d(\xi) = d^+(\xi) - d^-(\xi)$, we then have

$$\ddot{w}_r - c\dot{w}_r - d^-(\xi) w_r \le -d^+(\xi) w_r \le 0$$

An application of the Maximum principle gives us

$$w_r \equiv 0$$

Impossible. Therefore $\bar{r}$ does not exist, we can therefore decrease r to zero without making z touching v.

Now if

$$w(\xi) = z(\xi) - v(\xi) \ge 0 \qquad on\ (-a, a)$$

and $w(-a) > 0$, $w(a) > 0$, then by maximux principle, we have $w(\xi) > 0$, i.e. $z(\xi) > v(\xi)$, $\xi \in [-a,\, a]$. □

Lemma 0.9. *Suppose f satisfies i)-v) and if $\theta_1 \le \theta_2$ and if $(c_{\theta_1},\, v_{\theta_1})$ and $(c_{\theta_1,\, \theta_2})$ are solutions of*

$$\begin{cases} v''_{\theta_1} - c_{\theta_1} v'_{\theta_1} + f(\theta_1) = 0 \\ v''_{\theta_2} - c_{\theta_2} v'_{\theta_2} + f(\theta_2) = 0 \\ v_{\theta_i}(-\infty) = 0,\ v_{\theta_i}(+\infty) = 1, \quad i = 1, 2 \end{cases}$$

then $c_{\theta_1} \geq c_{\theta_2}$.

Proof. c_{θ_1}, c_{θ_2} are uniquely determined by Lemma1.

If $c_{\theta_1} < c_{\theta_2}$, then $-c_{\theta_1} > -c_{\theta_2}$ and by Lemma 1 again

$$\begin{aligned} v'_{\theta_1} &> 0, v'_{\theta_2} > 0 \quad on\ R \\ v_{\theta_1} &= H(\tfrac{1}{c_{\theta_1}}\xi) + \circ(1) \\ v_{\theta_2} &= H(\tfrac{1}{c_{\theta_2}}\xi) + \circ(1) \end{aligned}$$

for ξ big enough, here f_{θ_1} and f_{θ_2} agree for u near 1.

By

$$\frac{\partial v_\theta}{\partial c_\theta} = H'(\frac{1}{c_\theta}\xi)\xi\frac{-1}{(c_\theta)^2} < 0$$

we have

$$v_{\theta_1}(\xi) > v_{\theta_2}(\xi)$$

Now Take T big enough then $v_{\theta_1}(T) > v_{\theta_2}(T)$ and by Lemma 1,

$$\begin{aligned} v_{\theta_1}(\xi) &= Ae^{c_{\theta_1}\xi} + \circ(e^{c_{\theta_1}\xi}) \\ v_{\theta_2}(\xi) &= Be^{c_{\theta_2}\xi} + \circ(e^{c_{\theta_2}\xi}) \end{aligned}$$

as $\xi \to -\infty$, hence

$$v_{\theta_1}(-T) > v_{\theta_2}(-T)$$

Next, we can normalize $v_{\theta_1}(0) = v_{\theta_2}(0) = \frac{1}{2}$, then

$$\begin{aligned} v''_{\theta_1} - c_{\theta_1} v'_{\theta_1} + f(v_{\theta_1}) &\leq 0, \\ v''_{\theta_2} - c_{\theta_2} v'_{\theta_2} + f(v_{\theta_2}) &= 0 \end{aligned}$$

and

$$v''_{\theta_2} - c_{\theta_1} v'_{\theta_2} + f_{\theta_2}(v_{\theta_2}) \geq 0 \qquad on\ [-T,\ T]$$

with boundary conditions

$$v_{\theta_2}(-T) < v_{\theta_1}(-T) < v_{\theta_1}(\xi) = 0 \qquad on\ (-T,\ T]$$

$$v_{\theta_2}(\xi) < v_{\theta_1}(T) < v_{\theta_1}(T) = 0 \qquad on\ [-T,\ T)$$

Then Lemma 3 implies

$$v_{\theta_2}(\xi) < v_{\theta_1}(\xi) \qquad on\ (-T,\ T)$$

Contradicts with the normalization. □

Lemma 0.10. *c_θ of (???) has an upper bound independent on θ.*

Proof. By lemma 1,

$$v_\theta(\xi) = Ae^{c_\theta \xi} + \circ(e^{c_\theta \xi}) \qquad as\ \xi \to -\infty$$
$$v_\theta = H(\tfrac{1}{c_\theta}\xi) + \circ(1) \qquad as\ \xi \to -\infty$$

We consider the following

$$w'' - kw' + g(w) = 0$$

g is to be determined later.

We first construct w. Let w be a smooth, strictly increasing function of ξ, $0 < w < 1$, with $\dot{w} > 0$ and $w(0) = \frac{1}{2}$. Furthermore, $w = e^{\lambda' \xi}$ for $\xi \le -N$, N range and egnals $1 - e^{-\lambda' \xi}$ for $\xi \ge N$. λ',τ' fixed with $0 < C_{\frac{1}{4}} \le \lambda' < C_\theta$ and τ' be any positive number . Choose k so large that $-kw < 0$ for all ξ. Since $\dot{w} > 0$, we can choose $g > 0$ on $(0,1)$, for all ξ. Observe that for $\xi \le -N$, i.e $w \le e^{-\lambda' N}$, $g(w) = \lambda'(k - \lambda')w$; while for $w \ge 1 - e^{-\tau' N}$, $g(w) = \tau'(k + \tau')(1 - w)$ Making k large, we may therefore achieve that

$$g(w) \ge f(w)$$

for $0 \le w \le e^{-\lambda' N}$ and $w \ge 1 - e^{-\tau' N}$.

The remaining values of w, we have $-N \le \xi \le N$, there $\dot{w}$ is bounded away from zero. We may increase K further to make $g \ge f$ everywhere, we have

$$w \ge v_\theta \qquad in\ \mid x \mid \ge N$$

We have

$$c_\theta \le k$$

Suppose on the contrary, $c_\theta \ge k$ for some θ and at $\xi = \pm N$, i.e $c_\theta > k$ and $g \ge f$, then

$$w'' - c_\theta w' + f_\theta(w) < 0$$

Therefore, we would have, by maximum principle, $w > v_\theta$ on $(-N, N)$. By scaling, $w(0) = v_\theta(0) = \frac{1}{2}$, contradition.

Now we are ready to ripe our existence theorem. □

Theorem 0.11. *Suppose f satisfies conditions i)-v), then there exists $c^* > 0$ and a function $\phi(\xi)$, $\xi \in R$, such that*

$$\begin{cases} \phi'' - c^*\phi' + \quad f(\phi) = 0 \\ \phi(-\infty) = 0, \quad \phi(+\infty) = 1 \end{cases}$$

and $\phi'(\xi) > 0$ on R. The constant c^* is uniquely determined by (*).

Proof. Choosing a decreasing sequence of $\{\theta_i\}$, $i = 1, 2, 3, \cdots$, $\theta_i \to 0$ as $i \to \infty$, we have, by Lemma ?,

$$c^* = \lim_{\theta \to 0} c_\theta$$

Hence, we can find by local estimates, as $\theta_i \to 0$, u_{θ_i} converges to a solution u of

$$\begin{cases} u'' - c^*u' + \quad f(u) = 0 \\ u(-\infty) = 0, \quad u(+\infty) = 1 \end{cases}$$

Furthermore, $u'_{\theta_i} > 0$ implies $u' \geq 0$ by passing through a limit. By the convergence of $u_{\theta_i} \to 0$ as $\xi \to -\infty$ and $u_{\theta_i} \to 1$ as $\xi \to +\infty$, taking limit, we have $u \to 0$ as $\xi \to -\infty$ and $u \to 1$ as $\xi \to +\infty$. We can integrate (???) by parts to conclude that $c* > 0$. □

Theorem 0.12. *There is no solution of (???) for any* $0 < c < c^*$.

Proof. The proof is the same as that in [1]. □

Theorem 0.13. *Suppose the conditions of Theorem 1 are satisfied, then for every* $c > c^*$, c^* *is given by Theorem 1, then*

$$\begin{cases} u'' - cu' + & f(u) = 0 \\ u(-\infty) = 0, & u(+\infty) = 1 \end{cases}$$

has a solution.

Proof. Step 1, we denote the solution obtained in Theorem 1 as $(u^*,\, c^*)$ and we have, by Theorem 1, $(u^*)' > 0$ for $\xi \in R$. Hence, for any $c > c^*$, we have

$$(u^*)'' - c(u^*)' + f(u^*) < 0$$

i.e. u^* is a supersolution of

$$u'' - cu' + f(u) = 0$$

Also, for any $0 < h < 1$, $f(h) > 0$, then h is a subsolution of (???). Now fix a constant $a \geq 1$, we choose $h \leq u^*(-a)$, therefore, there exists a function v, such that

$$\begin{cases} v'' - cv' + & f(v) = 0 \\ v(-a) = h, & v(a) = u^*(a) \end{cases}$$

v maybe obtained by monotone iteration and we have $h \leq v \leq u^*$ on $[-a,\, a]$ and $v' > 0$ on $(-a,\, a)$.

Step 2. Now we shift u^* and let $u^{*r} = u^*(\xi + r)$ and $h^r = u^*(-a + r)$, then by step 1, there exists v^r, such that

$$\begin{cases} (v^r)'' - c(v^r)' + & f(v^r) = 0 \\ v^r(-a) = h, & v^r(a) = u*(a) \end{cases}$$

Uniqueness of the solution of the above equations implies v^r depending continuously on r. Since $(u^*)' > 0$, then v^r is a subsolution corresponding to any $r' > r$ and then we have $v^{r'} > v^r$, let $r \to \infty$, we have $v^r \to 1$ by $u^* \to 1$ and $v^r \to 0$ as $r \to -\infty$. Then , there exists some $r = \bar{r}$, such that $v^{\bar{r}}(0) = \frac{1}{2}$.

Fix $r = \bar{r}$ and denote the solution of

$$\begin{cases} (v^{\bar{r}})'' - c(v^{\bar{r}})' + f(v^{\bar{r}}) = 0, [-a,\, a] \\ 0 < v^{\bar{r}} < 1,\ (v^r)' > 0 \\ v^{\bar{r}}(0) = \frac{1}{2} \end{cases}$$

as u^a. □

For any $p > 1$, the family u^a is bounded by $W^{2,p}$ norm, as $a \to +\infty$, there exists a sequence $a_j \to +\infty$, such that $u^{a_j} \to u$ on any compact set of $[-a, a]$ uniformly and u satisfies

$$\begin{cases} u'' - cu' + & f(u) = 0 \\ u' \geq 0, & u(0) = \frac{1}{2} \end{cases}$$

Then, $u(\xi) \neq 0$, $u(\xi) \neq 1$, so $u(\xi)$ is not a constant and $u(-\infty) = 0$, $u(+\infty) = 1$.

Part 3. Asymptotic expansion

In the previous part, we proved the existence of the following equation

$$\begin{cases} u'' - cu' + & f(u) = 0 \\ u(-\infty) = 0, & u(+\infty) = 1 \end{cases}$$

and f satisfies conditions i)-v) and $c \geq c^*$ and (u, c) is a solution of (???).

Now, by

$$u'' - cu' + f(u) = 0,$$

we obtain

$$u''' - cu'' + f'(u)u' = 0,$$

then

$$\frac{u'''}{u'} - c\frac{u''}{u'} = -f'(u) \to 0 \qquad as\, \xi \to +\infty$$

Let

$$\begin{array}{l} \phi(\xi) = \frac{u''}{u'} \\ \phi'(\xi) = \frac{u'''u' - (u'')^2}{(u')^2} = \frac{u'''}{u'} - \phi^2 \end{array}$$

We have

$$\begin{array}{l} \phi'(\xi) - c\phi + \phi^2 \\ = \frac{u'''}{u'} - \phi^2 - c\phi + \phi^2 \\ = \frac{u'''}{u'} - c\frac{u''}{u'} \\ = -f'(u) \\ \to 0 \qquad as\, \xi \to -\infty \end{array}$$

We have,

Lemma 0.14. *As $\xi \to -\infty$, $\phi(\xi) \to \lambda$, λ satisfies $\lambda^2 - c\lambda = 0$ where $c > c^*$.*

Proof. First we show that $|\, \phi \,|$is bounded as $\xi \to -\infty$.

Assume that the above assertion is not true, then

first, if ϕ is monotone for some $(-\infty,\, \xi_0)$ with $|\xi_0|$ sufficiently large, then, $|\phi|$ is also monotone for some $(-\infty,\, \xi_1)$. Hence, $|\phi| \to \infty$ as $\xi \to -\infty$ and therefore, $\frac{1}{|\phi|} \to 0$ as $\xi \to -\infty$, then

$$\frac{\phi'}{\phi^2} - c\frac{1}{\phi} + 1 \to 0, \qquad as\, \xi \to -\infty$$

implies

$$\frac{\phi'}{\phi^2} \to -1 \qquad as\, \xi \to -\infty$$

i.e.

$$(\frac{1}{\phi})' \to 1 \qquad as\, \xi \to -\infty$$

Contradiction.

We then assume that ϕis not monotone for any interval $(-\infty, \xi_0)$, here ξ_0 is large and arbitrary. But then we can choose η_i as the local maximum of $|\phi|$ hence are the local maxima or local minima of $|\phi|$, $\phi'(\eta_i) = 0$ and $\eta_i \to -\infty$ and $|\phi(\eta_i)| = +\infty$, as $i \to \infty$.

By

$$\phi' - c\phi + \phi^2 \to 0 \qquad as\, \xi \to -\infty$$

we have

$$\phi'(\xi) = c\phi - \phi^2$$

then,

$$\phi'(\eta_i) = c\phi(\eta_i) - (\phi(\eta_i))^2 \to -\infty$$

Impossible.

Therefore, $|\phi|$ is bounded as $\xi \to -\infty$.

Next, we show that $\phi(\xi)$ tends to a finite number as $\xi \to -\infty$.

Suppose on the contrary,

$$\alpha \equiv \lim_{\xi \to -\infty} \sup \phi(\xi) > \lim_{\xi \to -\infty} \inf \phi(\xi) \equiv \beta$$

Then we have several cases to work with.

1, $\alpha \neq 0, \alpha \neq c$.

Let ξ_i be the local maxima of ϕ. Then by $\phi'(\xi_i) = 0$ and $\phi(\xi_i) \to \infty, i \to +\infty,$we have $\phi'(\xi_i) - c\phi(\tau_1) + \phi(\tau_1)^2 \to -c\alpha + \alpha^2 \neq 0$, contradiction.

2, $\alpha = 0$ or $\alpha = c$.

We only consider the case $\alpha = c$ but otherwise could be proved samilarly.

We choose ϵ small enough such that $\alpha - \epsilon \neq 0$, pick up sequence $\{\xi_i\}$, $i = 1, 2, 3 \cdots$such that $\phi(\xi_i) = \alpha - \frac{\epsilon}{2}$ and $\phi'(\xi) \leq 0$ (we can choosea subsequence if necessary). Then,

$$\phi'(\xi_i) - c\phi(\xi_i) + (\phi(\xi_i))^2 \leq (\alpha - \frac{\epsilon}{2})(\alpha - \frac{\epsilon}{2} - c) < 0$$

contradiction.

Hence $\phi(x)$ has a finite limit as $\xi \to -\infty$. Suppose $\phi(\xi) \to \alpha$ as $\xi \to -\infty$, then $-c\alpha + \alpha^2 = 0$.

Now we need to show

$$\frac{u''}{u'} \to 0 \qquad as\, \xi \to -\infty.$$

If not, we would have

$$\frac{u''}{u'} = c$$

then u decays exponentially, This is in contradiction with the results of [4].

Therefore,

$$u'' = \circ(u') \qquad as\, \xi \to -\infty$$

Similarly, as $\xi \to +\infty$, we also have

$$u'' = \circ(u')$$

□

Then the asymptotic behaviours of the travelling solution can be derived exactly the same way as in Theorem 1.

The only case remainning is the asymptotic expansion of u^* as $\xi \to -\infty$. Consider

$$\begin{cases} (u^*)'' - c(u^*)' + & f(u^*) = 0 \\ u^*(-\infty) = 0, & u^*(+\infty) = 1 \end{cases}$$

and $c* = \lim_{i\to\infty} c_{\theta_i}$ and $\lim_{i\to\infty} \theta_i = 0$, $(u^*)' > 0$.

We have

$$(u^*)'' - c(u^*)' = -f(u^*) < 0$$

i.e.

$$(u^*)'' < c^*(u^*)'$$

Now we prove that

$$(u^*)' > \frac{c^*}{2} u^*.$$

By our construction of u^*, $u^* = \lim_{i\to\infty} v_{\theta_i}$, where v_{θ_i} is the solution of (???) with $\theta = \theta_i$. By assumption i), ii), we have, for u is close to 0, there exists constant $k > 0$ and $p > 1$, such that

$$f(u) \le k\, u^p$$

hence,

$$0 = (u^*)'' - c(u^*)' + f(u^*) \le (u^*)'' - c(u^*)' + k(u^*)^p$$

because $u* \to 0$ as $\xi \to -\infty$, we have

$$(u^*)'' - c(u^*)' + \delta u^* \geq 0$$

for some $\delta > 0$ small and $\xi \to -\infty$.

Now since $f_{\theta_i} \equiv 0$ for $0 < u < \theta_i$, i.e, for some interval $\xi \in (-\infty,\ \xi_i)$, we have

$$(v_{\theta_i})'' - c_{\theta_i} v'_{\theta_i} = 0$$

as $\xi \in (-\infty,\ \xi_i)$.

Hence $(v_{\theta_i})'' - \frac{c_{\theta_i}}{2} v'_{\theta_i} \geq 0\ in\,(-\infty,\ \xi_i)$. We obtain $(v_{\theta_i})' - \frac{c_{\theta_i}}{2} v_{\theta_i} \geq 0\ in\,(-\infty,\ \xi_i)$.

Now let

$$R_i = \sup\left\{R \in (-\infty, 0] \mid v'_i > \frac{c_{\theta_i}}{2} v_i, \quad for\ all\ \xi \leq R\right\}$$

.

We have $R_i \geq \xi_i$, we need to show

$$R_i \equiv 0$$

Otherwise, $R_i < 0$ then we would have

$$\left\{ \begin{array}{l} v'_{\theta_i}(\xi) \geq \frac{c_{\theta_i}}{2} v_{\theta_i}(\xi) \\ v''_{\theta_i} - \frac{c_{\theta_i}}{2} v'_{\theta_i} + \delta v_{\theta_i} \geq 0 \end{array} \right. in\,(-\infty,\ R_i]$$

then

$$v''_{\theta_i} - \frac{c_{\theta_i}}{2} v'_{\theta_i} + \frac{2\delta}{c_{\theta_i}} v'_{\theta_i} \geq 0$$

Integrating from $-\infty$ to ξ, we would have

$$v'_{\theta_i} \geq (\frac{c_{\theta_i}}{2} - \frac{2\delta}{c_{\theta_i}}) v_{\theta_i} > \frac{c_{\theta_i}}{4} v_{\theta_i} \quad in\,(-\infty,\ R_i]$$

as long as δ is chosen small enough. Contradiction with our definition of R_i. Hence, by (1), (2) and taking limit, we have for some $D > 0$,

$$u^*(\xi) \leq De^{c^*\xi}$$

We now sum up our result in this part.

Theorem 0.15. *The travelling wave solution of system (???) has the following asymptoti behavoirs*

1. $c = c^*$.

$$\begin{array}{ll} \phi(\xi) = H(\frac{1}{c^*}\xi) + \circ(1) & as\ \xi \to +\infty; \\ \phi(\xi) = Ae^{c^*\xi} + \circ(e^{c^*\xi}) & as\ \xi \to -\infty. \end{array}$$

2. $c > c^*$,

We have the same expansion as the first expansion but with c^* *replaced by* c.

Proof. See above and Theorem 1. ☐

Here is a corollary with can be used to our example.

Corollary 0.16. *Assume that in addition to conditions i)-v) f satisfies that* $f''(1) > 0$ *and* $f''(0) < 0$, *and* f *is of* $C^{3,\alpha}$*,then the travelling wave solution of (???) has the following asymptotic expansions:*

$$\begin{array}{ll} \phi(\xi) = 1 - \frac{2c^*}{f''(1)}\frac{1}{\xi} + \circ(\frac{1}{\xi}) & as\ \xi \to \infty; \\ \phi(\xi) = Ae^{c^*\xi} + \circ(e^{c^*\xi}) & as\ \xi \to -\infty. \end{array}$$

when $c = c^*$.

Proof. The asymptotic expansion for ϕ at $-\infty$ is just in Theorem ???. We now derive the asymptotic expansion for ϕ at $+\infty$. We will do it by construct super and sub-solutions.

Consider function

$$v(\xi) = 1 - \frac{2c^*}{f''(1)}\frac{1}{\xi} + \frac{B\ln\xi}{\xi^2}$$

where the constant B is to be determined later.

$$\begin{array}{ll} v'(\xi) & = \frac{2c^*}{f''(1)}\frac{1}{\xi^2} + \frac{B-2B\ln\xi}{\xi^3} \\ v''(\xi) & = -\frac{4c^*}{\xi^3} - \frac{5B-6B\ln\xi}{\xi^4} \end{array}$$

After a tedious computation, We reach that

$$\begin{array}{ll} \hat{L}v & = v'' - c^*v + f(v) \\ & = \frac{1}{\xi^4}(-Bc^*\xi - \frac{4c^*}{f''(1)} - \frac{f'''(1)}{6}(\frac{2c^*}{f''(1)})^3\xi + \circ(1)) \end{array}$$

Now, if $B = B_1 > 0$ is sufficiently large, $\hat{L}v < 0$, for ξ large enough, then $v_1(\xi) = 1 - \frac{2c^*}{f''(1)}\frac{1}{\xi} + \frac{B_1\ln\xi}{\xi^2}$ is super-solution of $\hat{L}v = 0$, similarly, we can choose $B_2 < 0$ and $|B_2|$ big such that $v_1(\xi) = 1 - \frac{2c^*}{f''(1)}\frac{1}{\xi} + \frac{B_1\ln\xi}{\xi^2}$ is a subsolution of $\hat{L}v = 0$. And of course, we have $v_2 < v_1$ for ξ large enough. Then , there exists $T > 0$, such that

$$0 < v_2(T) < v_1(T) < 1$$

and at the same time, we can shift our ϕ, such that

$$0 < v_2(T) < \phi(T) < v_1(T) < 1$$

We then obtain that

$$\begin{cases} \hat{L}v_2 > \hat{L}\phi > \hat{L}v_1, \\ v_2(T) < v_2(T) < v_1(T), \\ v_2(+\infty) = v_1(+\infty) = \phi(+\infty) = 1 \end{cases}$$

and

$$\begin{array}{ll} \hat{L}(v_2 - \phi) & = (v_2'' - \phi'') - c^*(v_2' - \phi') + f(v_2) - f(\phi) \\ & \doteq w'' - c^*w' + f'(\cdot)w > 0 \end{array}$$

By Maximum Prinple, we see $v_2 < \phi$, in thet same way, we have $v_1 > \phi$ for $\xi > 0$ sufficiently large. Therefore

$$\phi(\xi) = 1 - \frac{2c^*}{f''(1)}\frac{1}{\xi} + \circ(\frac{1}{\xi}) \qquad as\, \xi \to \infty$$

□

Applications:

We consider the Josephson Junction equation

$$\ddot{x} + \sigma\dot{x} + \sin x - 1 = 0$$

Reversing time and space variables by $t \to -t$, $x \to -x$, we have

$$\ddot{x} - \sigma\dot{x} + 1 + \sin x = 0 \tag{0.4}$$

Therefore, corresponding to the notions of Theorem 1, we have

$$f(x) = 1 + \sin x \geq 0 x \in [-\frac{\pi}{2},\, \frac{3\pi}{2}]$$

and 1. $f(-\frac{\pi}{2}) = f(\frac{3\pi}{2}) = 0$;

2. $f'(-\frac{\pi}{2}) = f'(\frac{3\pi}{2}) = 0$;

3. f is Lipschtz continuous on $[-\frac{\pi}{2},\, \frac{3\pi}{2}]$;

4.

$f'(x) = \cos x \quad satisfies$

$f'(x) > 0, \qquad for\, x\, close\, to - \frac{\pi}{2}$

$f'(x) < 0 \qquad for\, x\, close\, to\, \frac{3\pi}{2}$

5. $f(x) = 1 + \sin x > 0 \qquad on\, (-\frac{\pi}{2},\, \frac{3\pi}{2})$

Hence, by Theorem 1, there exsits $\sigma = \sigma^* > 0$ and a solution (x^*, σ^*) of (J,J) connecting the equilibrium $(-\frac{\pi}{2},\, 0)$ and $(\frac{3\pi}{2},\, 0)$ and $\frac{dx^*}{dt} > 0$, for $t \in R$; for $\sigma < \sigma^*$, (J,J) has no solutions, for $\sigma > \sigma^*$, (J,J) has solution $(u,\, \sigma)$. This result is in consistence with that of [5]. It is easy to see that the connection has the asymptotic expansion described by the Corollary.

Part 4. Stability

Proof. In this part, we prove that the travelling wave solution $\phi(\xi)$, $\xi \in R$, derived from last section, of system □

$$\begin{cases} u_t = & u_{xx} + f(u) \\ u(0) & = \bar{u} \end{cases} \tag{0.5}$$

is asymptotically stable in the Lyapunov's sense.

To do so, we first introduce some definitions.

Definition 0.17. We say that the travelling wave solution $\phi(\xi, c)$, $\xi \in R, c > 0$ is asymptotically stable if for any solution $u(\xi,\, t)$ of (1) with $u(\xi, 0) = \bar{u}$, we have $\| \bar{u}(\xi) - \phi(\xi) \|_E < \delta$, $\delta > 0$ is a small real number, implies

$$\| u(\xi, t) - \phi(\xi, t) \|_E < Ae^{-bt}$$

Here $b > 0$ is a constant, $A > 0$ and $t > 0$, E is some Banach space.

Definition 0.18. If L is a linear operator in a Banach space, a normal point for L is any complex number which isa in the resolvent set of L or is an isolated eigenvalue of L of finite multiplicity. Any other complex number is in the essential spectrum. (see Gohberg and Krein [9], see also [14]).

We also have

Definition 0.19. We define space C as

$$C = \{u \mid u\, is\, continuous\, and\, bounded\, function\}$$

and the norm on C is

$$\| u \|_C = \sup_{x\in R} |u(x)|.$$

Space C_{σ_1,σ_2} (weighted Banach space) is definde as

$$C_{\sigma_1,\sigma_2} = \{u \in C | \lim_{|x|\to\infty} u(x)(e^{\sigma_1 x} + e^{-\sigma_2 x}) = 0\}$$

with norm

$$\| u \|_{C_{\sigma_1,\sigma_2}} = \sup_{x\in R} |u(x)(e^{\sigma_1 x} + e^{-\sigma_2 x})|.$$

here we require σ_1, σ_2 to be nonnegative real numbers.

Similarly, we can define $C^2_{\sigma_1,\sigma_2}$ and C^2_0 as well, for example,

$$C^2_{\sigma_1,\sigma_2} = \left\{u \mid u \in C_{\sigma_1,\sigma_2},\ u' \in C_{\sigma_1,\sigma_2},\ u'' \in C_{\sigma_1,\sigma_2}\right\}$$

with norm

$$|| u ||_{C^2_{\sigma_{1,2}}} = \sup || \frac{d^i u}{dx^i} \cdot (e^{\sigma_1 x} + e^{\sigma_2 x}) ||$$

we can also introduce complex $C^1\sigma_1, \sigma_2,\ C^2\sigma_1, \sigma_2 \cdots$ in the same fashion.

Now introduce transformation

$$u(\xi, t) = \phi(\xi, c) + v(\xi, t) \tag{0.6}$$

then (1) turns into

$$\begin{cases} v_t = v_\xi - cv_\xi + f'(\phi)v + R(v, \phi) \\ v(0) = u(0) - \phi(\xi, c) \end{cases} \tag{0.7}$$

Here $c > 0$ is the speed of wave ϕ and

$$R(v, \phi) = f(\phi + v) - f(\phi) - f'(\phi)v \tag{0.8}$$

is the nonlinear term.

Definition 0.20. Denote

$$Lv = v_{\xi\xi} - cv_\xi + f'(\phi)v \tag{0.9}$$

we are going to study the spectrum of L in various Banach spaces, we have

Lemma 0.21. *Consider operator*

$$Lu = u'' + A(x)u' + B(x)u \quad x \in R$$

with $A(x),\ B(x) \to a_\pm,\ b_\pm$ *as* $x \to \pm\infty$,

$$\sigma_e(L) \subset \left\{ \lambda \,|\, Re\,\lambda - \frac{Im\,\lambda}{a_\pm^2} \le -b_\pm \right\}$$

In particular,

$$\sup Re\,\sigma(L) = \max(-b_+, -b_-).$$

Proof: see [9] page 140.

Then, in space C, whe have that the essential spectrum of operator L defined by (5) is contained in the region bounded by curve

$$\Gamma = -\xi^2 - c_i\xi = 0, \qquad \xi \in R$$

and $\sup Re\,\sigma_e(L) = 0$. Hence, the continuous spectrum of operator L touches the origin in the complex plane. In order to obtain any asymptotic stability result for system (1), we need consider the operator L in some weighted Banach space.

Let opertor T defined as

$$T : C_{\sigma_1,\sigma_2} \to C_0, \qquad Tu = (e^{\sigma_1 x} + e^{-\sigma_2 x})u$$

then it is easy to see that T has the following properties:

1. T is linear and bounded operator.
2. T has a bounded inverse, which we denote as $T^{-1} : C_0 \to C_\sigma$ and $T^{-1}u = (e^{\sigma_1 x} + e^{-\sigma_2 x})^{-1}u,\ u \in C_0$.

Now consider the operator

$$\tilde{L} = TLT^{-1}u$$

and $\tilde{L} : c_0 \to c_0$with domain $D(\tilde{L}) = C_2$.

We have

$$\tilde{L}u = u'' - (2g_1(x) + c)u' + (cg_1(x) - g_2(x) - 2g_3(x) + f'(\phi))v$$

where

$$g_1(x) = \frac{\sigma_1 e^{\sigma_1 x} - \sigma_2 e^{\sigma_2 x}}{e^{\sigma_1 x} + e^{-\sigma_2 x}}$$
$$g_2(x) = \frac{\sigma_1^2 e^{\sigma_1 x} - \sigma_2^2 e^{\sigma_2 x}}{e^{\sigma_1 x} + e^{-\sigma_2 x}}$$
$$g_3(x) = \frac{\sigma_1 e^{\sigma_1 x} - \sigma_2 e^{-\sigma_2 x}}{(e^{\sigma_1 x} + e^{-\sigma_2 x})^2}$$

with

$$\lim_{x\to-\infty} g_1(x) = -\sigma_2, \qquad \lim_{x\to+\infty} g_1(x) = -\sigma_1;$$
$$\lim_{x\to-\infty} g_2(x) = \sigma_2^2, \qquad \lim_{x\to+\infty} g_2(x) = \sigma_2^2;$$
$$\lim_{|x|\to\infty} g_3(x) = 0$$

The operator $\tilde{L}$ at "infinity" are

$$\tilde{L}^+v = v'' - (2\sigma_1 + c)v' + (-\sigma_1^2 + c\sigma_1)v \qquad at + \infty;$$
$$\tilde{L}^-v = v'' - (-2\sigma_2 + c)v' + (-\sigma_2^2 + c\sigma_2)v \qquad at - \infty.$$

Lemma 0.22. *The essential spectrum of operator* $\tilde{L}$ *in space* C *is contained in some closed angle in the left half complex plane for any* $\sigma_1 > C$ *and* $\sigma_2 > 0$ *and there are only discrete spectrum of* $\tilde{L}$ *left ouside this angle.*

Proof. By lemma 1, the essential spectrum of $\tilde{L}$ is contained in the region bounded by curves

$$\lambda_+ = -\xi^2 - (2\sigma_1 + c)_i\xi + (-\sigma^2 + c\sigma_1)$$

and

$$\lambda_- = -\xi^2 - (-2\sigma_2 + c)_i\xi + (-\sigma^2 + c\sigma_2)$$

where ξ is a paremeter, $\xi \in R$.

For any $\xi \in R$,

$$\begin{aligned} -\xi^2 + (-\sigma_1^2 + c\sigma_1), \\ -\xi^2 + (-\sigma_2^2 + c\sigma_2) \end{aligned}$$

are negative iff $\sigma_1 > C$ and $\sigma_2 > 0$ and in turn, $\sigma_1 > C$, $\sigma_2 > 0$ implies the essential spectrum of $\tilde{L}$ are in the left half complex plane. As $|\xi| \to \infty$, $\lambda+$, $\lambda-$ are asymptotically parabolas, therefore, we can find an angle with vertex at the negative horizontal axis bounding those essential spectrum of $\tilde{L}$ in space C. Outside this angle we still have (if any) some isolated eigenvalues of $\tilde{L}$. □

Lemma 0.23. *Consider operator L in space C_{σ_1,σ_2}, then then essential spectrum of L is contained in some closed angle in the left complext plane.*

Proof. By relation 6, we see that the normal point of $\tilde{L}$ in C and L in C_{σ_1,σ_2} coincide ...

Next, we prove that the operator L in space C_{σ_1,σ_2} has no eigenvalues with nonegative real parts. We proceed as follows:

Introducing transformation

$$u(x) = v(x)e^{\frac{c}{2}x}$$

for operator

$$Lu = u'' - cu' + f'(\phi)u$$

we have

$$Mv = v'' + [-\frac{c^2}{4} + f'(\phi)]v$$

we consider operator L in the weighted Banach space $C_0, \frac{c}{2}$with weight $w = 1+e^{-\frac{c}{2}x}$ and $C_{0,\frac{c}{2}} = \left\{u \in C \,|\, \sup_{x\in R} |\,(1+e^{-\frac{c}{2}x})\,u(x)\,| < \infty\right\}$and we consider M in space C .

Operators L and M have relation

$$L = e^{\frac{c}{2}x}\,M\,e^{-\frac{c}{2}x}$$

We have □

Lemma 0.24. *The essential spectrum of M is contained in the region $\left\{\lambda \in C \mid Re\,\lambda \le -\frac{c^2}{4}\right\}$ out of this region, M has discrete eigenvalues which are real numbers. M does not have positive eigenvalues.*

Proof. By lemma 1, the essential spectrum of M is contained in the region $\left\{\lambda \in C \,|\, Re\,\lambda + \frac{c^2}{4} \le 0\right\}$and outside this region M has only isolated eigenvalues with finite multiplications. By ?, M is self-adjoint in space C , hence the eigenvalues of M are real.

Now we prove that M deos not have positive eigenvalues. Before doing so, we first study the asymptotic behaviour of the solution of

$$M\,u = \lambda\,u \quad with\ \ Re\,\lambda > -\frac{c^2}{4}$$

we have

$$\lambda'' - (\frac{c^2}{4} + \lambda + f'(\phi))\,\lambda = 0 \tag{0.10}$$

then

as $x \to +\infty$, (10) ha two independent solutions

$$\begin{aligned}
\psi_1^+ &= e^{-\sqrt{\lambda+\frac{c^2}{4}}x} x^{-\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}}[1+\circ(x^{-\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}})];\\
(\psi_1^+)' &= e^{-\sqrt{\lambda+\frac{c^2}{4}}x} x^{-\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}}[\frac{-c}{\sqrt{\lambda+\frac{c^2}{4}}}+\circ(1)];\\
\psi_2^+ &= e^{\sqrt{\lambda+\frac{c^2}{4}}x} x^{+\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}}[1+\phi(x^{+\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}})];\\
(\psi_2^+)' &= e^{\sqrt{\lambda+\frac{c^2}{4}}x} x^{+\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}}[\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}+\circ(1)];
\end{aligned}$$

and as $x \to -\infty$,

$$\begin{aligned}
\psi_1^- &= e^{-\sqrt{\lambda+\frac{c^2}{4}}x}(1+\circ(\tfrac{1}{|\lambda|}));\\
(\psi_1^-)' &= e^{\sqrt{\lambda+\frac{c^2}{4}}x}(\sqrt{\lambda+\tfrac{c^2}{4}}+\circ(1));\\
\psi_2^- &= e^{-\sqrt{\lambda+\frac{c^2}{4}}x}(1+\circ(\tfrac{1}{|\lambda|}));\\
(\psi_2^-)' &= e^{\sqrt{\lambda+\frac{c^2}{4}}x}(\sqrt{\lambda+\tfrac{c^2}{4}}+\circ(1));
\end{aligned}$$

Proof: The last assertion can be proved exactly the same way as in [6].

Now we prove the first part.

For ψ_1^+, we introduce transformation

$$\psi = e^{-\sqrt{\lambda+\frac{c^2}{4}}x} z_1$$

then

$$z'' - 2\sqrt{\lambda+\frac{c^2}{4}}z_1' + f'(\phi)z_1 = 0$$

By our previous section, we have

$$f'(\phi) = f'(1) + f''(1)(\phi-1) + \frac{f''(1)}{2!}(\phi-1)^2 + \circ((\phi-1)^2)$$

$$\begin{aligned}
f'(\phi) &= f'(1) + f''(1)(\phi-1) + \tfrac{f''(1)}{2!}(\phi-1)^2 + \circ((\phi-1)^2)\\
&= f''(1)(\phi-1) + \tfrac{f'''(1)}{2}(\phi-1)^2 + \circ((\phi-1)^2)
\end{aligned} \tag{0.11}$$

and

$$\phi = 1 - \tfrac{2c}{f''(1)}\tfrac{1}{x} + \circ(x^{-1}) \qquad as\ \ x \to +\infty \tag{0.12}$$

we have

$$z_1 = x^{-\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}}[1+\circ(x^{-\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}})]$$

Returning to ψ, we have

$$\psi_1^+ = e^{-\sqrt{\lambda+\frac{c^2}{4}}x}x^{-\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}}[1+\circ(x^{-\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}})]$$

Next, we consider $v = \phi' e^{-\frac{c}{2}x}$, ϕ is the travelling wave solution, then bu the asymptotic expansion of ϕ,

$$v \in C$$

and also we have

$$L\phi' = (\phi')'' - c(\phi')' + f'(\phi)\phi' = 0$$

then

$$MV = 0$$

For any solution of

$$(M-\lambda)\psi = 0$$

with $\lambda > 0$, at $+\infty$, by lemma 5

$$\psi \sim e^{-\sqrt{\lambda+\frac{c^2}{4}}x}x^{-\sqrt{\lambda+\frac{c^2}{4}}}[1+\circ(\frac{1}{|\lambda|})]$$

at $+\infty$

$$\psi \sim e^{\sqrt{\lambda+\frac{c^2}{4}}x}[1+\circ(\frac{1}{|\lambda|})]$$

then for

$$u = \frac{\psi}{v}$$

we have as $x \to +\infty$,

$$u \sim e^{\frac{c}{2}-\sqrt{\lambda+\frac{c^2}{4}}x}x^{-\sqrt{\lambda+\frac{c^2}{4}}}/\phi'$$

as $x \to -\infty$,

$$u \sim e^{c-\sqrt{\lambda+\frac{c^2}{4}}x}$$

Therefor, $u \to 0$ as $x \to \pm\infty$, and $u > 0$ for $x \in R$ but u satisfies

$$u'' - 2\frac{\psi'}{\psi}u + \lambda u = 0$$

By maximum priciple

$$u \equiv 0 \qquad \forall x \in R$$

Impossible. Hence, M does not have positive eigenvalues,and also,0 is an eigenvalue of M in C with eigenfunction $v = \phi' e^{-\frac{c}{2}x}$.

We discuss the spectrum of L in space $C_{0,\frac{c}{2}}$. We need the following: □

Lemma 0.25. $\| (\lambda - M)^{-1}u \|_{C_2} \leq \frac{C}{|\lambda|} \| u \|_C$, *where* $\lambda \in \left\{\lambda \in | Re\lambda - \frac{c^2}{4} > 0\right\}$.

Proof. As it was proven in [], we need to estimate

$$\int_{\infty}^{\infty} \mid G(x,\, y,\, \lambda) \mid dy$$

where

$$G(x,y,\lambda) = \frac{1}{w(\lambda)} \begin{cases} \psi_1^-(y,\,\lambda)\psi_1^+(x,\,\lambda), & x > y; \\ \psi_1^+(y,\,\lambda)\psi_1^-(x,\,\lambda), & x < y. \end{cases}$$

and $w(\lambda)$ is the uronskian

$$w(\lambda) = \begin{vmatrix} \psi_1^+ & \psi_1^- \\ (\psi_1^+)' & (\psi_1^-)' \end{vmatrix} = 2\sqrt{\lambda + \frac{c^2}{4}} + \circ(1)$$

as $\lambda \to +\infty$.

$$\int_{-\infty}^{+\infty} |G(x,y,\lambda)|\, dy \le \frac{1}{w(\lambda)}[\int_{-\infty}^{x} \mid \psi_1^+(x,\lambda) \mid\mid \psi_1^-(y,\lambda) \mid dy + \int_{x}^{+\infty} \mid \psi_1^-(x,\lambda) \mid\mid \psi_1^+(y,\lambda) \mid dy]$$

Now we suppose $x > 0$ and first estimate

$$\begin{aligned} &\int_{-\infty}^{x} \mid \psi_1^+(x,\lambda) \mid\mid \psi_1^-(y,\lambda) \mid dy \\ &= \int_{-\infty}^{0} \mid \psi_1^+(x,\lambda) \mid\mid \psi_1^-(y,\lambda) \mid dy + \int_0^x \mid \psi_1^+(x,\lambda) \mid\mid \psi_1^-(y,\lambda) \mid dy \\ &\le c_1 e^{-\sqrt{\lambda+\frac{c^2}{4}}x} x^{-\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}} \int_{-\infty}^{0} e^{\sqrt{\lambda+\frac{c^2}{4}}y} dy \\ &+ c_2 \int_0^x e^{-\sqrt{\lambda+\frac{c^2}{4}}x} x^{-\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}} e^{\sqrt{\lambda+\frac{c^2}{4}}y} y^{\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}} \\ &\le c_3 e^{-\sqrt{\lambda+\frac{c^2}{4}}x} e^{-\sqrt{\lambda+\frac{c^2}{4}}} [\frac{1}{\sqrt{\lambda+\frac{c^2}{4}}} + \circ(1)] \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^{x} \mid \psi_1^+(x,\lambda) \mid\mid \psi_1^-(y,\lambda) \mid dy &= \int_{-\infty}^{0} \mid \psi_1^+(x,\lambda) \mid\mid \psi_1^-(y,\lambda) \mid dy \\ &+ \int_0^x \mid \psi_1^+(x,\lambda) \mid\mid \psi_1^-(y,\lambda) \mid dy \\ &\le c_1 e^{-\sqrt{\lambda+\frac{c^2}{4}}x} x^{-\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}} \int_{-\infty}^{0} e^{\sqrt{\lambda+\frac{c^2}{4}}y} dy \\ &+ c_2 \int_0^x e^{-\sqrt{\lambda+\frac{c^2}{4}}x} x^{-\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}} e^{\sqrt{\lambda+\frac{c^2}{4}}y} y^{\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}} \\ &\le c_3 e^{-\sqrt{\lambda+\frac{c^2}{4}}x} e^{-\sqrt{\lambda+\frac{c^2}{4}}} [\frac{1}{\sqrt{\lambda+\frac{c^2}{4}}} + \circ(1)] \end{aligned}$$

and

$$\begin{aligned} &\int_x^{\infty} \mid \phi_1(y,\lambda) \mid\mid \psi_,(x,\lambda) \mid dy \\ &\le c_4 e^{\sqrt{\lambda+\frac{c^2}{4}}x} x^{\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}} \int_x^{\infty} e^{-\sqrt{\lambda+\frac{c^2}{4}}y} y^{-\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}} dy \\ &\le c_4 e^{\sqrt{\lambda+\frac{c^2}{4}}x} x^{\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}} x^{-\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}} \int_x^{\infty} e^{-\sqrt{\lambda+\frac{c^2}{4}}y} dy \\ &c_4 e^{\sqrt{\lambda+\frac{c^2}{4}}x} (\frac{1}{\sqrt{\lambda+\frac{c^2}{4}}} e^{-\sqrt{\lambda+\frac{c^2}{4}}x}) \\ &\le \frac{c'(\delta)}{\sqrt{(\lambda)}} \end{aligned}$$

Hence by (???), (???),(???)

$$\begin{aligned} &\int_{-\infty}^{\infty} \mid G(x,\, y,\, \lambda) \mid dy \\ &\le \frac{c(\delta)}{|\lambda|} \end{aligned}$$

where δ is the closed angle.

Similarly, for $x < 0$, the estimate is the same. □

Next, we prove that the resolvent of L in $C_{0,\frac{c}{2}} and$ that of M coincide in the region

$$P = \left\{ c \in C \mid Re\sqrt{\lambda + \frac{c^2}{4}} > \frac{c}{2} \right\}$$

Lemma 0.26. *The resolvent of L and that of M coincide in the region P.*

Proof. We estimate

$$\begin{aligned} &\| (\lambda - L)^{-1} \|_{C_{0,\frac{c}{2}}} \\ =&\| (\lambda - L)^{-1} u(1 + e^{-\frac{c}{2}x}) \|_c \\ =&\| e^{\frac{c}{2}x}(\lambda - M)^{-1} e^{-\frac{c}{2}x} u + e^{\frac{c}{2}x}(\lambda - M)^{-1} e^{-\frac{c}{2}x} u^{-\frac{c}{2}x} \| \\ \leq&\| e^{\frac{c}{2}x}(\lambda - M)^{-1} e^{-\frac{c}{2}x} u(x) \|_c + \| (\lambda - M)^{-1} e^{-\frac{c}{2}x} u \|_c \end{aligned}$$

We require $u \in C_{0,\frac{c}{2}}$, then $e^{-\frac{c}{2}x} u \in C$.

By lemma 5, the second term in the last inequality is bounded, so we just estimate the first term.

$$\begin{aligned} &\| e^{\frac{c}{2}x}(\lambda - M)^{-1} e^{-\frac{c}{2}x} u(x) \| \leq e^{\frac{c}{2}x} \int_{-\infty}^{\infty} \mid G(x, y, \lambda) \mid e^{-\frac{c}{2}y} \mid u(y) \mid dy \\ &\leq e^{\frac{c}{2}x} \frac{1}{|w(\lambda)} [\int_{-\infty}^{x} \mid \psi_1^+(x, \lambda) \mid \psi_1^-(y, \lambda) \mid e^{-\frac{c}{2}y} \mid u(y) \mid dy + \int_x^{+\infty} \mid \psi_1^+(y, \lambda) \|\, \psi_1^-(y, \lambda) \mid e^{-\frac{c}{2}y} \mid u(y) \mid dy \end{aligned}$$

first, as in lemma 5, we let

$x > 0$,

$$\begin{aligned} &\int_{-\infty}^{x} \mid \psi_1^+(x, \lambda) \mid \psi_1^-(y, \lambda) \mid e^{-\frac{c}{2}y} \mid u(y) \mid dy \\ &\leq c_1 e^{-\sqrt{\lambda+\frac{c^2}{4}}x} x^{-\sqrt{\lambda+\frac{c^2}{4}}} [\int_{-\infty}^{0} e^{-\sqrt{\lambda+\frac{c^2}{4}}y} dy + c_2 \int_0^x e^{\sqrt{\lambda+\frac{c^2}{4}}y} y^{\overline{\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}}} e^{-\frac{c}{2}y} dy] \| u \|_{C_{0,\frac{c}{2}}} \\ &\leq c_1 e^{-\sqrt{\lambda+\frac{c^2}{4}}x} x^{-\sqrt{\lambda+\frac{c^2}{4}}} [\frac{1}{Re\sqrt{\lambda+\frac{c^2}{4}}} + c_2 x^{\overline{\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}}} \int_0^x e^{(-\frac{c}{2}+\sqrt{\lambda+\frac{c^2}{4}})y} dy] \| u \|_{C_{0,\frac{c}{2}}} \\ &\leq c_3 e^{-\sqrt{\lambda+\frac{c^2}{4}}x} [\frac{1}{Re\sqrt{\lambda+\frac{c^2}{4}}} + C_4 \frac{e^{-(\frac{c}{2}-\sqrt{\lambda+\frac{c^2}{4}})x}}{Re\sqrt{\lambda+\frac{c^2}{4}} - \frac{c}{2}}] \| u \|_{C_{0,\frac{c}{2}}} \end{aligned}$$

and

$$\begin{aligned} &\int_x^{+\infty} \mid \psi_1^+(y, \lambda) \|\, \psi_1^-(y, \lambda) \mid e^{-\frac{c}{2}y} \mid u(y) \mid dy \\ &\leq c_5 e^{+\sqrt{\lambda+\frac{c^2}{4}}x} x^{\sqrt{\lambda+\frac{c^2}{4}}} [\int_x^{+\infty} e^{-\sqrt{\lambda+\frac{c^2}{4}}y} y^{-\overline{\frac{c}{\sqrt{\lambda+\frac{c^2}{4}}}}} e^{-\frac{c}{2}y} dy] \| u \| dy \\ &\leq c_5 e^{Re\sqrt{\lambda+\frac{c^2}{4}}x} x^{\sqrt{\lambda+\frac{c^2}{4}}} x^{-\sqrt{\lambda+\frac{c^2}{4}}} \int_x^{+\infty} e^{(-\frac{c}{2}-\sqrt{\lambda+\frac{c^2}{4}})y} \mid u(y) \mid dy \\ &\leq c_5 e^{Re\sqrt{\lambda+\frac{c^2}{4}}x} \frac{e^{-(\frac{c}{2}-Re\sqrt{\lambda+\frac{c^2}{4}})y}}{Re\sqrt{\lambda+\frac{c^2}{4}} + \frac{c}{2}} \| u \|_{C_{0,\frac{c}{2}}} \end{aligned}$$

Therefore, by (???) and (???), the expression (???) is bounded by

$$\frac{C_6(\delta)}{\mid \lambda \mid} \| u \| C_{0,\frac{c}{2}} as$$

as $x \to \infty$ and $\mid \lambda \mid \to \infty$ in the closed angle provided $\lambda \in P$. □

Lemma 0.27. *The operator L, considered in C_{σ_1,σ_2}, where $\sigma_1 > c$, $\sigma_2 = \frac{c}{2}$has no eigenvalues with nonegative real parts.*

Proof. We denote L in space C_{σ_1,σ_2}, as L_{σ_1,σ_2} and L in space $C_{0,\frac{c}{2}}$as $L_{C_{0,\frac{c}{2}}}$, in space C as L_C.

We have the following relations: □

1. $C_{\sigma_1,\sigma_2} \subseteq C_{0,\frac{c}{2}} \subseteq C$

2. if u is a solution of

$$Lu = \lambda u \qquad in \quad C_{\sigma_1,\sigma_2}$$

then u is a solution of

$$Lu = \lambda u \qquad in \quad C_{0,\frac{c}{2}}$$

Therefore,

$$\sigma_p(L_{C_{\tau_1,\tau_2}}) \subset \sigma_p(L_{C_{0,\frac{c}{2}}})$$

here $\sigma_p(L)$ denote the eigenvalues of L.

By lemma 6, $L_{C_{0,\frac{c}{2}}}$ does not have eigenvalues with positive real pars, we then conclude that $L_{C_{\sigma_1,\sigma_2}}$ does not have eigenvalues with positive real pars and also, it is easy to see, C_{σ_1,σ_2} does not have 0 as its eigenvalue, by lemma ?- lemma ?, $L_{C_{\sigma_1,\sigma_2}}$ does not have eigenvalues with zero or positive real parts.

We next show that

$$(\lambda - L)^{-1}$$

is a generator of an anlytic semigroup.

Lemma 0.28. *Operator $L_{C_{\sigma_1,\sigma_2}}$ has a dense domain of definition. For $Re\,\lambda > 0$ and $\mid \lambda \mid$ is sufficiently large, $(\lambda - L)^{-1}$ exists and is defined on all of C_{σ_1,σ_2} and satisfies the following estimate*

$$\| (\lambda - L)^{-1} \|_{C_{\sigma_1,\sigma_2}} \leq \frac{c}{1+ \mid \lambda \mid}$$

where C is a positive constant.

Proof. The proof is basically the same as that of [12].. Now we state our stability Theorem. □

Theorem 0.29. *The travelling wave solution $\phi(\xi)$ is asymptotic stable in space C_{σ_1,σ_2}.*

Proof. See [12]. □

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