Generalized averages for solutions of two-point Dirichlet problems

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Abstract

For very general two-point boundary value problems we show that any positive solution satisfies a certain integral relation. As a consequence we obtain some new uniqueness and multiplicity results.

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1 Introduction

We show that any positive solution of very general two-point Dirichlet boundary value problems satisfies a certain integral relation. To explain our approach as well as the title of the present paper, we consider an example, which was a part of a problem proposal in P. Korman [2]. It is known that the boundary-value problem

$$(1.1) \qquad \begin{aligned} u'' + u^3 = 0 \quad \text{for} \quad 0 < x < L \\ u(0) = u(L) = 0 \end{aligned}$$

has a unique positive solution, $u(x) > 0$ for $x \in (0, L)$ (as can be seen by e.g. phase-plane analysis). We claim that for any $L > 0$ this solution satisfies

$$\int_0^L u(x)\, dx = \frac{\pi}{\sqrt{2}}. \tag{1.2}$$

This formula gives us the average of the solution $u(x)$. To establish (1.2) we proceed following [1], [4]. We set $y = \int_0^x u(s)\, ds$, and then $h(y) = u^2$. The problem (1.1) transforms to

$$\begin{aligned} h'' + 2h = 0 \quad \text{for} \quad 0 < y < R \\ h(0) = 0 \\ h(R) = 0, \end{aligned} \tag{1.3}$$

where

$$R = \int_0^L u(s)\, ds. \tag{1.4}$$

Solution of (1.3) satisfying $h(0) = 0$ is

$$h(y) = c \sin \sqrt{2}\, y, \ \ c \text{ - an arbitrary constant.}$$

Condition $h(R) = 0$ combined with the positivity of $h(y)$ implies that $\sqrt{2}\, R = \pi$, and by (1.4) the proof follows.

We show that the above procedure can be generalized to any positive $f(u)$. The resulting integral relation implies certain uniqueness and multiplicity results. Another possible application is for verification of validity of numerical methods, or it might be even possible to incorporate the integral relation into the numerical method itself (similarly to the way energy is used for symplectic integrators).

2 An integral relation with applications to uniqueness and multiplicity

We now consider a general problem

$$u'' + f(u) = 0 \quad \text{for } 0 < x < L, \ \ u(0) = u(L) = 0. \tag{2.1}$$

We define the functions $l(u)$ and $g(u)$ as follows

$$l(u) = \left(\int_0^u f(t)\, dt \right)^{\frac{1}{2}}, \ \ g(u) = l'(u). \tag{2.2}$$

Theorem 2.1 *Assume that $f(u) \in C^1(R_+)$. Let $u(x)$ be a positive classical solution of (2.1). Assume that for all u in the range of $u(x)$ we have*

$$f(u) > 0. \tag{2.3}$$

Then for any L we have

$$\int_0^L g(u(x))\,dx = \frac{\pi}{\sqrt{2}}. \tag{2.4}$$

Proof: Condition (2.3) of course implies that the functions $l(u) \in C^2(R_+)$ and $g(u) \in C^1(R_+)$ are defined, and $g(u) > 0$ for $u > 0$. Similarly to [1] and [4], we change the independent variable, substituting

$$y = \int_0^x g(u(s))\,ds. \tag{2.5}$$

Notice that $\frac{dy}{dx} > 0$, and hence the inverse function $x = x(y)$ is defined. We then may regard the solution u alternatively as a function of y, with $u(y) = u(x(y))$, or as a function of x, $u(x) = u(y(x))$. Observe that $y(x)$ is of class C^2, and hence the same is true of $x(y)$. It follows that $u(y)$ is of class C^2. The equation in (2.1) transforms as follows

$$u'' + f(u) = u_{yy} g^2 + u_y^2 g' g + f(u) = 0. \tag{2.6}$$

Since $l' = \dfrac{1}{2}\dfrac{f}{l}$, we further transform (2.6)

$$l'(u)\left(u_{yy} l'(u) + u_y^2 l''(u) + 2l(u)\right) = 0. \tag{2.7}$$

We now set

$$h(y) \equiv l(u(y)).$$

By the above remarks $h(y)$ is twice continuously differentiable. Then (2.7) takes the form

$$l'(u)\,(h'' + 2h) = 0. \tag{2.8}$$

Since $l'(u) > 0$ it follows that the equation

$$h'' + 2h = 0, \tag{2.9}$$

holds on the interval $(0, R)$, where $R = \int_0^L g(u(s))\,ds > 0$. The boundary conditions are of course

$$h(0) = h(R) = 0.$$

Since $h(y)$ is positive on $(0, R)$ we conclude the integral relation (2.4) the same way we did in the introduction.

The following lemma is known. We present its proof for completeness.

Lemma 2.1 *Any two positive solutions $u(x)$ and $v(x)$ of (2.1) are strictly ordered, i.e. we may assume that*

$$u(x) < v(x) \quad \text{for all } x \in (0, L). \tag{2.10}$$

Proof: By uniqueness for initial-value problems any positive solution of (2.1) is symmetric with respect to any critical point. It follows that any solution is an even function relatively to $x = \frac{L}{2}$, and $u'(x) < 0$ on $(\frac{L}{2}, L)$. Assuming the lemma to be false, let $\xi \in (\frac{L}{2}, L)$ be the largest point where u and v intersect and assume for definiteness that (2.10) holds on (ξ, L). We now multiply (2.1) by u' and integrate over (ξ, L). Then we multiply the same equation for $v(x)$ by v', integrate over (ξ, L), and finally subtract the results, obtaining

$$\frac{1}{2}u'^2(L) - \frac{1}{2}v'^2(L) + \frac{1}{2}v'^2(\xi) - \frac{1}{2}u'^2(\xi) = 0. \tag{2.11}$$

We have a contradiction, since we have two negative differences on the left in (2.11).

Theorem 2.2 *Assume that the function $f(u) \in C^1(R_+)$ satisfies (2.3) and either*

$$-\frac{1}{2}f^2(u) + f'(u)\int_0^u f(s)\,ds > 0 \quad \text{for almost all } u > 0 \tag{2.12}$$

or the opposite inequality holds. Then the problem (2.1) has at most one positive solution.

Proof: Conditions (2.12) and (2.3) imply that $g' = l'' > 0$ (or < 0), and hence the function $g(u)$ is either strictly increasing or strictly decreasing. In view of Theorem 2.1 and Lemma 2.1 the result follows.

Example 1. Assume that the function $f(u) \in C^2(R_+)$ satisfies

$$\text{either} \quad f(0) > 0, \quad \text{or} \quad f(0) = 0 \text{ and } f'(0) > 0,$$

and also

$$f''(u) < 0 \quad \text{for all } u > 0.$$

Then the problem (2.1) has at most one positive solution.

Indeed, if $f(u) > 0$ for all $u > 0$ then condition (2.3) holds trivially. Otherwise, $f(u)$ has exactly one root, say $\bar{u} > 0$, and then (2.3) holds for $0 < u < \bar{u}$, but this is an inequality that any positive solution satisfies. Denoting by $\psi(u)$ the quantity on the left in (2.12), we see that $\psi(0) < 0$ and $\psi'(u) < 0$ for all $u > 0$, and hence $\psi(u) < 0$ for all $u > 0$, and the Theorem 2.2 applies.

Arguing similarly we establish the next example.

Example 2. Assume that the function $f(u) \in C^2(R_+)$ satisfies

$$f(0) = 0, \;\; f'(0) \geq 0, \quad \text{and } f''(u) > 0 \text{ for all } u > 0.$$

Then the problem (2.1) has at most one positive solution.

Example 3. Consider the problem

$$u'' + \lambda(e^u - a) = 0 \;\text{ on } (0, L)\;, \;\; u(0) = u(L) = 0, \tag{2.13}$$

with a positive parameter λ and a constant $a > 0$. If $a = 1$ then by the previous example the problem (2.13) has at most one positive solution for any $\lambda > 0$ (actually one can show that there is exactly one positive solution for $0 < \lambda < \frac{\pi^2}{L^2}$ and no positive solutions for $\lambda \geq \frac{\pi^2}{L^2}$). If we fix $0 < a < 1$, then using methods of bifurcation theory, see e.g., [3], one shows existence of a critical $\lambda_0 > 0$, so that the problem (2.13) has exactly two, one or zero solutions, depending on whether $\lambda < \lambda_0$, $\lambda = \lambda_0$ or $\lambda > \lambda_0$.

The last example shows that our uniqueness result, Theorem 2.2, is in a sense precise, as it picks up a borderline case for uniqueness. We remark that the results given in the above examples are more or less known. However, the condition (2.12) of Theorem 2.2 is much more general. It allows one to considerably relax the requirements of concavity (convexity) in the above examples.

We now turn to multiplicity of positive solutions.

Theorem 2.3 *Assume that the function $f(u) \in C^2(R_+)$ satisfies (2.3) and*

$$\frac{1}{2} f''(u) \left(\int_0^u f(s)\, ds \right)^2 + \frac{3}{8} f^3(u) - \frac{3}{4} f(u) f'(u) \int_0^u f(s)\, ds > 0 \tag{2.14}$$

for all $u > 0$. Then the problem (2.1) admits at most two positive solutions.

Proof: By a direct computation we see that (2.14) is equivalent to

$$g''(u) > 0 \quad \text{for all } u > 0. \tag{2.15}$$

Assume on the contrary that there are three solutions u, v and w. By Lemma 2.1 they are strictly ordered, i.e. we may assume that

$$u(x) < v(x) < w(x) \quad \text{for all } x \in (0, L).$$

Set $p(x) = v(x) - u(x)$, $q(x) = w(x) - u(x)$. Clearly

$$0 < p(x) < q(x) \quad \text{for all } x \in (0, L). \tag{2.16}$$

Writing the formula (2.4) at u and v respectively, and then subtracting, we conclude (using the mean-value theorem)

$$\int_0^L \int_0^1 g'\left(\theta v(x) + (1-\theta)u(x)\right) p(x)\, d\theta dx = 0. \tag{2.17}$$

Similarly

$$\int_0^L \int_0^1 g'\left(\theta w(x) + (1-\theta)u(x)\right) q(x)\, d\theta dx = 0. \tag{2.18}$$

In view of the above inequalities, the integrand in (2.18) is pointwise greater than the one in (2.17), a contradiction.

We compare our multiplicity result with the following well-known theorem, whose simple proof we include for completeness.

Theorem 2.4 *Assume that the function $f(u) \in C^2(R_+)$ and either*

$$f''(u) > 0 \quad \textit{for all } u > 0 \tag{2.19}$$

or the opposite inequality holds. Then the problem (2.1) admits at most two positive solutions.

Proof: Assuming existence of three solutions $u(x) < v(x) < w(x)$, and introducing $0 < p(x) < q(x)$ as above, we obtain by subtracting the corresponding equations

$$p'' + c(x)p = 0 \ \text{ for } x \in (0, L), \ p(0) = p(L) = 0, \tag{2.20}$$

with $c(x) = \displaystyle\int_0^1 f'\left(\theta v + (1-\theta)u\right)\, d\theta$. Similarly,

$$q'' + d(x)q = 0 \ \text{ for } x \in (0, L), \ q(0) = q(L) = 0, \tag{2.21}$$

with $d(x) = \int_0^1 f'(\theta w + (1-\theta)u)\, d\theta$. Since $d(x) > c(x)$, we obtain a contradiction by the Sturm's comparison theorem.

Even though our condition (2.14) does not imply the known condition (2.19), we do obtain an extension of the Theorem 2.4, allowing nonlinearities which are neither convex or concave, as the following example shows.

Example. Consider $f(u) = 1 - bu^2 + 5u^4$ with a constant $b > 0$. Clearly f is concave for $0 < u < \sqrt{\frac{b}{30}}$ and convex for $u > \sqrt{\frac{b}{30}}$. If, say, $b < 1$, then f is positive for all $u > 0$, i.e. the condition (2.3) holds. To verify the condition (2.14) it is convenient to use its equivalent form (2.15), i.e. $l'''(u) = g''(u) > 0$. A computer algebra computation gives $l'''(u)$ as a fraction, whose denominator is $\left(\int_0^u f(s)\, ds\right)^{\frac{5}{2}} > 0$, and the numerator can be written as a difference of two polynomials $p_1(u) - p_2(u)$, where

$$p_1(u) = 15u^{12} - 15u^8 + 165u^4 + 3,$$

and $p_2(u)$ is a polynomial of degree 10, all of whose coefficients go to zero with b. Clearly

$$p_1(u) > u^{12} \quad \text{for } u > 1.$$

By choosing b small we can achieve $p_2(u) < p_1(u)$ on $[0,1]$, and then decreasing b, if necessary, we can obtain $p_2(u) < u^{10} < u^{12}$ for $u > 1$. This implies that $p_2(u) < p_1(u)$ for all $u \geq 0$, i.e. $l'''(u) > 0$, and the Theorem 2.3 applies.

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