EXISTENCE AND NONEXISTENCE OF POSITIVE SINGULAR SOLUTIONS FOR SEMILINEAR ELLIPTIC PROBLEMS WITH APPLICATIONS IN ASTROPHYSICS.

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1. INTRODUCTION

Stationary radially symmetric models in stellar dynamics have been studied extensively. Eddington [ED] in 1915 introduced the equation

$$\Delta u + \frac{e^{2u}}{1+|x|^2} = 0 \text{ in } \mathbf{R}^3$$

to study the gravitational potential u of a globular cluster of stars. Fifteen years later, Matukuma [M1,M2] proposed the equation

$$\Delta u + \frac{u^p}{1+|x|^2} = 0 \text{ in } \mathbf{R}^3$$

to improve Eddington's model. Here $u > 0$ represents the gravitational potential, $\rho = (4\pi)^{-1}(1+|x|^2)^{-1}u^p$ is the density and $\int\limits_{\mathbf{R}^3} \rho dx$ represents the total mass. (See also Ni and Yotsutani [NY]). In 1972, Peebles [P1,P2] gives for the first time a derivation of the steady-state distribution of stars near a massive collapsed object, such as a black hole, located at the center of a globular cluster. The same year, Peebles [P1] motivated the observer and theoretician with the title of his paper, "Black holes are where you find them" and concluded, that "there can be no conclusions until we find a black hole". Since then, a great deal has been written about black holes by astrophysicists (see the recent review article by Shapiro [SH]). However, the question of the existence of a black hole in a globular cluster is still open. Hubble Space Telescope (HST) observations of globular cluster cores should improve the observational basis for confirming or denying the presence of massive black holes in globular clusters (see; e.g., Cohen [CO]). Core collapse does occur; for instance, using the HST, Bendinelli et.al. [BCD, May 1993] have documented the first detection of a collapsed core globular cluster in M31. It is also probable that M15 is in a current state of collapse [K].

From the theoretical point of view, it is of interest to prove the existence of solutions of appropriate models for black holes. For a relativistic model this was first done (unintentionally) by Schwarzschild [SC] in 1916 within a month of Einstein's publications of the theory of general relativity. However, neither Einstein nor Schwarzschild knew at that time that Schwarzschild's solutions contained a complete description of a black hole. A very recent study of the existence of black hole solutions for the Einstein -Yang/Mills equations is due to Smoller, Wasserman, and Yau [SWY].

This paper is concerned with the existence of *"black hole solutions"* of two different models; i.e., Matukuma's, as well, Hénon's-type equations. In contrast to known models for black holes located at the center of globular clusters, we do not assume the existence of a black hole, instead we impose a restriction on the model which a black hole must satisfy; i.e., the gravitational potential of the cluster behaves like $1/r (r = |x|)$ near the center. We then study the existence and nonexistence of solutions of the model.

To study the existence of stationary radially symmetric solutions of the standard system involving the Boltzmann's equation, it is sufficient to solve an equation of the form

$$\Delta u + h(|x|, u) = 0.$$

(See; e.g., Batt et al. [BFH, p. 170]).

Motivated by the above remarks we consider the n-dimensional version of

$$(1.1) \begin{cases} \Delta u + A(|x|)u^p = 0 & \text{in } \mathbf{R}^3 \backslash \{0\} \\ u > 0 & \text{in } \mathbf{R}^3 \backslash \{0\} \\ u \sim 1/|x| & \text{near } \quad x = 0 \\ u \sim 1/|x| & \text{at } \quad \infty, \end{cases}$$

where $p > 1$ and A is nonnegative and locally Hölder continuous on $(0, \infty)$. The presence of A indicates that the model is not necessarily isotropic. It follows from our result in $\mathbf{R}^n$ (Theorem 2.1 below) that *(1.1) has infinitely many solutions with finite total mass provided*

$$\int_0^\infty r^{2-p} A(r) dr < \infty. \tag{1.2}$$

We will use the super-subsolution approach as in Ni [N2], Naito [NA] and Kawano[KA] to obtain the result. Consequently, our result can be extended to include nonradial $A = A(x)$, if we add conditions like

$$\int_0^\infty r^{2-p}\big(\sup_{|x|=r} A(x)\big)dr < \infty.$$

Our main result applied to the Matukuma's equation implies that (1.1) with $A(r) = 1/(1+r^2)$ has infinitely many radial solutions with finite total mass provided $1 < p < 3$ (Corollary 2.2 below). This is in contrast to the uniqueness result for an entire solution in $\mathbf{R}^n$ obtained recently by Yanagida [YA] and Kawano, Yanagida, and Yotsutani [KYY].

Furthermore, we shall observer that the stellar density profile $\rho(r)$ for (1.1) with the Matukuma's equation satisfies, near the center,

$$\rho(r) \sim \frac{r^{-(m+3/2)}}{4\pi(1+r^2)} \tag{$*$}$$

and

$$\frac{c_1}{r(1+r^2)} < \rho(r) < \frac{c_2}{r^3(1+r^2)},$$

for some $c_1, c_2 > 0$ and $-1/2 < m < 3/2$. (cf. Ni and Yutsutani [NY, p. 30]). We point out that the above inequality is valid for $r > 0$ sufficiently small. In a similar model, for r away from zero ($0 < r_{\mathrm{D}} \le r \le r_a$), Bahcall and Wolf [BW] obtained the density profile $r^{-7/4}$. It is interesting to note the density we obtain in $(*)$ with $m = 1/2$ resembles that of Lightman and Shapiro [LS], where r_{D} is "extremely small" and $0 < r_{\mathrm{D}} \le r \le r_a$.

We shall further observe that the integrability condition in (1.2) is sharp (Theorem 2.3 below). Consequently, we obtain that (1.1) with the Matukuma's equation has no solution provided $p \ge 3$ (Corollary 2.4 below).

Singular positive solutions for elliptic problems in $\mathbf{R}^n$ related to (1.1) have been studied recently by Ni and Serrin [NS2], Bandle and Marcus [BM], Serrin [S1,S2], Guedda and Veron [GV], Schoen [S], Gidas and Spruck [GS], Johnson, Pan and Yi [JPY], and Caffarelli, Gidas and Spruck [CGS]. The main feature in our results is the prescribed behavior of solutions near the origin.

It is also well known (see; e.g., Ni [N3]) that

$$(1.3)\begin{cases} \Delta u + |x|^{-\ell}u^p = 0 & \text{in } \ \Omega\backslash\{0\} \\ u > 0 & \text{in } \ \Omega\backslash\{0\}, u = 0 \ \text{ on } \ \partial\Omega \\ u \to \infty & \text{as } \ |x| \to 0, \end{cases}$$

where Ω is an open ball in $\mathbf{R}^n$ containing the origin, has infinitely many singular radial solutions provided $1 < p < (n + 2 - 2\ell)/(n - 2)$ and $\ell < 2$. If u is a C^2 solution of (1.3) with $1 < p < (n - \ell)/(n - 2)$ and a nonremovable singularity at $x = 0$, then there exist positive constants C_1 and C_2 such that

$$\frac{C_1}{|x|^{n-2}} \le u(x) \le \frac{C_2}{|x|^{n-2}} \quad \text{near} \qquad x = 0, \tag{1.4}$$

and singular solutions are not distribution solutions at $x = 0$. (Gidas and Spruck [GS]).

We shall generalize the above existence result for solutions satisfying (1.4). More specifically we consider

$$(1.5)\begin{cases} \Delta u + f(|x|, u) = 0 & \text{in } \ B_{r_0}\backslash\{0\} \\ u > 0 & \text{in } \ B_{r_0}\backslash\{0\} \\ u = 0 & \text{on } \ \partial B_{r_0} \end{cases}$$

where f is continuous on $(0, r_0] \times [0, \infty)$, and B_{r_0} is the ball centered at 0 and radius r_0 in $\mathbf{R}^n (n \ge 3)$. When $n = 3$, (1.5) includes the special case $f(|x|, u) = |x|^{-\ell}u^p$, where u represents the gravitational potential of a stationary rotating stellar system. (Hénon [H]). It follows from our result (Theorem A below) that *(1.5) in this situation has infinitely many (radial) solutions behaving like* $1/|x|$ *near the origin, provided* $1 < p < 3 - \ell$. *Furthermore, there are no such solutions if* $p \ge 3 - \ell$ (Corollary 3.6 and Theorem 3.8 below).

The explicit dependence on $|x|$ of the nonlinearity in (1.5) is of interest for applications to real clusters in which the velocity distribution may not be isotropic (cf. Hénon [H , p. 233]).

Recent works for singular solutions in bounded domains include Ni [N1,N3], Senba, Ebihara and Furusho [SEF], Caffarelli, Gidas and Spruck [CGS], Aviles [A], Brezis and Oswald [BO], and Lions [L].

Our main result concerning (1.5) is

Theorem A. *Let $f(r,u) \geq 0$ for $u \geq 0$ and $r \in (0, r_0]$. Assume there exists a function $\bar{f}(r,s)$, nondecreasing in $s \geq 0$, for each $r \in (0, r_0]$, such that $f(r,s) \leq s\bar{f}(r,s)$ for $s \geq 0, r \in (0, r_0]$. Moreover, suppose there exists $\delta > 0$ with*

$$d \equiv (n-2)^{-1} \int_0^{r_0} r\bar{f}(r, \delta r^{2-n}) dr < 1.$$

Then for each $\alpha \in (0, \delta]$ there exists a radial solution u of (1.5) with finite mass, such that

$$\alpha(1-d)\varphi(r) \leq u(r) \leq \alpha\varphi(r), \quad 0 < r \leq r_0,$$

where $\varphi(r) = r^{2-n} - r_0^{2-n}$. Furthermore, $u'(r_0) = -(\alpha r_0^{1-n})(n-2)$.

We point out that the inequality $d < 1$ in Theorem A is sharp in some sense (Theorem 3.5 below). Theorem A applied to the Matukuma's equation implies that (1.5) with $f(|x|, u) = u^p/(1+|x|^2)$ in $B_{r_0} \backslash \{0\}$ has infinitely many radial solutions behaving like r^{2-n} near $r = 0$, provided $1 < p < n/(n-2)$.

Let A be continuous on $(0, r_0]$ with $0 \leq A(r) = O(r^{-\ell})$ near $r = 0, \ell < 2$. As a corollary of Theorem A we also have that

$$(1.6) \begin{cases} \Delta u + A(r)u^p = 0 & \text{in } B_{r_0} \backslash \{0\} \\ u > 0 & \text{in } B_{r_0} \backslash \{0\} \\ u = 0 & \text{on } \partial B_{r_0} \\ u \sim r^{2-n} & \text{near } r = 0 \end{cases}$$

has infinitely many radial solutions provided $1 < p < (n-\ell)/(n-2)$. (See also Corollary 3.2 below).

If $\ell < -1$ and $p \geq (n-\ell)/(n-2)$, it follows from a known result (see; e.g., Ni and Sacks [NS1]) that (1.6) has no solution in $C^2(B_{r_0}\backslash\{0\})$. We provide (Theorem 3.8 below) a proof of the case $-1 \leq \ell < 2,\ p \geq (n-\ell)/(n-2)$. Other nonexistence results have been obtained by Aviles [A], Ni [N1,N3] and Gidas and Spruck [GS] .

We further prove a nonexistence result for the equation in (1.6) with no restriction at the origin. More specifically we have

Theorem B. *Let* $A:(0,\infty)\longrightarrow[0,\infty)$ *be continuous with*

$$r^{-\eta}A\left(\frac{1}{r}\right)$$

nondecreasing where $\eta \equiv \frac{n+2-p(n-2)}{2}$. *Then for* $p>1$,

$$(1.7)\begin{cases}\Delta u + A(r)u^p = 0 & \textit{in}\ \ B_{r_0}\backslash\{0\}\\ u>0 & \textit{in}\ \ B_{r_0}\backslash\{0\}\\ u=0 & \textit{on}\ \partial B_{r_0}\end{cases}$$

has no radial solution.

The interesting feature of (1.7) is that no behavior of u at the origin is required. Another general nonexistence result of this type can be easily obtained from Lemma 1.1 by Ni and Sacks [NS1].

It follows from Theorem B and Ni [N3] that (1.7) with $A(r)=r^{-\ell}$ has no radial solution provided $p\geq(n+2-2\ell)/(n-2), \ell<2$. When $\ell=0$ the result is due to Ni and Serrin [NS2].

§2. SINGULAR POSITIVE SOLUTIONS FOR MATUKUMA'S TYPE EQUATION IN $\mathbf{R^n}\backslash\{\mathbf{0}\}$.

Let $p>1$ and $n\geq3$. In this section we consider the problem

$$(2.1)\begin{cases}\Delta u + A(|x|)u^p = 0 & \text{in}\ \ \mathbf{R}^n\backslash\{0\}\\ u>0 & \text{in}\ \ \mathbf{R}^n\backslash\{0\}\\ u\sim r^{2-n} & \text{near}\ \ r=0\\ u\sim r^{2-n} & \text{at}\ \ \ \infty\end{cases}$$

Theorem 2.1. *Let $A : (0, \infty) \longrightarrow [0, \infty)$ be a locally Hölder continuous function such that*

$$\int_0^\infty r^{n-1-p(n-2)} A(r) dr < \infty. \tag{2.2}$$

Then (2.1) has infinitely many solutions with finite total mass.

Proof. Similar to the work by Li and Ni [LN], we will make the following Kelvin-transform. Let

$$y = x/|x|^2, \ V(x) = |x|^{2-n} u(y) = |y|^{n-2} u(y)$$

and the desired singularity at $y = 0$ is guaranteed if $V(x) \longrightarrow C(> 0)$ as $|x| \longrightarrow \infty$.

V satisfies

$$\Delta V + |x|^{-n-2+p(n-2)} A(1/|x|) V^p = 0 \text{ in } \mathbf{R}^n \backslash \{0\}. \tag{2.3}$$

The result is then an immediate consequence of the following claim. There exists a $\beta > 0$ such that for any constant $C \in (0, \beta)$, there exists a positive solution V of (2.3) in $\mathbf{R}^n$ with $V \in C(\mathbf{R}^n) \cap C^2(\mathbf{R}^n \backslash \{0\}), V(x) \longrightarrow C$ as $|x| \longrightarrow \infty$. The proof of this assertion is essentially an adoption of results by Ni [N2], Naito [NA], Kawano [KA], and Ni and Yotsutani [NY].

Let

$$\bar{A} \equiv \int_0^\infty r^{n-1-p(n-2)} A(r) dr$$

and

$$\beta = \min$$

$$\Bigg\{$$

$\Bigg($

$\text{(n-2)}/\bar{A}\Big)^{1/(p-1)}, (1-1/p)(\frac{1}{p^{1/(p-1)}})(\frac{n-2}{\bar{A}})^{\frac{1}{p-1}}\Big\}. Considerg(t) = t - \frac{\bar{A}}{n-2}t^p. Thenforeach$ C $\in (0,\beta)$, there exists $\alpha > 0$ such that

$$C = \alpha - \frac{\bar{A}}{n-2}\alpha^p \text{ and } C - \frac{\bar{A}}{n-2}C^p > 0.$$

Let $W(r) = \alpha - \frac{\alpha^p}{r^{n-2}}\int\limits_0^r s^{n-3}\Bigg(\int\limits_0^s t^{p(n-2)-n-1}A(\frac{1}{t})dt\Bigg)ds$

and $h(s) = \int\limits_0^s t^{p(n-2)-n-1}A(\frac{1}{t})dt$. Then

$$= \alpha - \frac{\alpha^p}{n-2}h(r) + \frac{\alpha^p}{n-2}\int\limits_0^\infty (\frac{s}{r})^{n-2}\chi_{[o,r]}(s)s^{p(n-2)-n-1}A(\frac{1}{s})ds \longrightarrow \alpha - \frac{\alpha^p}{n-2}\bar{A} = C \qquad (r \longrightarrow$$

$\infty) bytheLebesgueDominatedConvergenceTheorem.Now,$

$$\begin{aligned} W'(r) &= \frac{\alpha^p}{r}\left[\frac{n-2}{r^{n-2}}\int\limits_0^r s^{n-3}h(s)ds - h(r)\right] \\ &= \frac{\alpha^p}{r}\left[\frac{n-2}{r^{n-2}}\int\limits_0^r s^{n-3}(h(s)-h(r))ds\right] \le 0 \end{aligned}$$

with

$$\Big(r^{n-1}W'(r)\Big)' = -\alpha^p r^{n-2}h'(r) = -\alpha^p r^{p(n-2)-3}A(\frac{1}{r}).$$

Thus W satisfies

$$\begin{cases} W'' + \frac{n-1}{r}W' + \alpha^p r^{p(n-2)-n-2}A(\frac{1}{r}) = 0 & \text{in } (0,\infty) \\ W(0) = \alpha > 0 \qquad , \ W \downarrow C & \text{at } \infty. \end{cases}$$

Since $\alpha \ge W \ge C$, we have

$$W'' + \frac{n-1}{r}W' + r^{p(n-2)-n-2}A(\frac{1}{r})W^p \le W'' + \frac{n-1}{r}W' + r^{p(n-2)-n-2}A(\frac{1}{r})\alpha^p = 0.$$

It is obvious that $\widetilde{W} \equiv C$ is a subsolution of (2.3). Thus a pair of super-subsolutions of (2.3) is obtained. Hence there exists a solution V of (2.3) which satisfies $C \le V \le W$

and

$$V \in C(\mathbf{R}^n) \cap C^2(\mathbf{R}^n \backslash \{0\}).$$

Finally, to show that the solutions obtained above have finite total mass we use the Kelvin transform introduced at the beginning of the proof and the fact that $dy = |x|^{-2n}dx$. We obtain

$$\begin{aligned} \int\limits_{\mathbf{R}^n\backslash\{0\}} A(y)u^p(y)dy &= \int\limits_{\mathbf{R}^n} A(1/|x|)(|x|^{n-2}V(x))^p|x|^{-2n}dx \\ &\le \alpha^p \int\limits_0^\infty r^{n-1}A(\frac{1}{r})r^{p(n-2)-2n}dr \\ &= \alpha^p \int\limits_0^\infty r^{n-1-p(n-2)}A(r)dr < \infty \end{aligned}$$

This completes the proof of the theorem.

REMARK 2.1. Asymptotic expansion of solutions of (2.1) near $x = 0$ can be obtained for suitable A's for which one can find an expansion at ∞ of solutions of (2.3) via the formulas in Li and Ni [LN]. For instance, this can be done for (2.1) with the Matukuma's equation.

REMARK 2.2. The solutions obtained by Naito in [NA] are radially symmetric about the origin. And radial symmetry of the solutions of (2.1), including Matukuma's equation, can be obtained via the techniques in [CL,LN, L2]. For instance, if $A(r)$ is nonincresing and $r^{(n-2)(1-p)}A(r) = O(r^{-2})$ at infinity, then all solutions of (2.1) must be radially symmetric. This follows from the works by [CL, LN, L2].

REMARK 2.3. It is easy to show that the solutions obtained in theorem 2.1 have infinite energy:

$$\int\limits_{\mathbf{R}^n} |\bigtriangledown u|^2 dx = \infty.$$

Corollary 2.2. *Let A be as above with $A(r) = O(r^{-\ell})$ at ∞, $A(r) = O(r^{-\sigma})$ at 0, for some $0 \le \sigma < \ell$ and*

$$max\ \{1,\ \frac{n-\ell}{n-2}\}\ < p < \frac{n-\sigma}{n-2},$$

Then (2.1) has infinitely many radial solutions with finite total mass.

As a simple consequence of this corollary ($\sigma = 0,\ \ell = 2$) we obtain that (2.1) with the Matukuma's equation, i.e.,

$$(2.4)\ \begin{cases} \Delta u + \frac{u^p}{1+|x|^2} = 0 & \text{in } \mathbf{R}^n\backslash\{0\} \\ u > 0 & \text{in } \mathbf{R}^n\backslash\{0\}, \\ u \sim r^{2-n} & \text{near} \quad x = 0, \\ u \sim r^{2-n} & \text{at} \quad \infty. \end{cases}$$

has infinitely many radial solutions with finite total mass provided $1 < p < n/(n-2)$.

Next we show that the integrability condition (2.2) in Theorem 2.1 is sharp in the following sense.

<u>**THEOREM 2.3**</u>. *Let* $A : (0,\infty) \longrightarrow [0,\infty)$ *be continuous with*

$$\int_0^\infty r^{n-1-p(n-2)} A(r) dr = \infty. \tag{2.5}$$

Then (2.1) has no radial solutions.

<u>**PROOF**</u>. Suppose that (2.1) has a radial solution u. Then as in the proof of Theorem 2.1, the Kelvin transform V of u satisfies

$$\Delta V + |x|^{p(n-2)-n-2} A(1/|x|) V^p = 0 \ \text{ in } \ \mathbf{R}^n\backslash\{0\}. \tag{2.3}$$

It suffices to show that (2.3) has no bounded positive radial solutions which are bounded away from zero at ∞. To this end, we apply Theorem 4.1 by Kawano [KA] with a minor modification. The result implies that if (2.3) has a positive bounded solution which is bounded away from zero, then

$$\begin{aligned} \infty > & \int_0^\infty r r^{p(n-2)-n-2} A(1/r) dr \\ = & \int_0^\infty r^{n-1-p(n-2)} A(r) dr, \end{aligned}$$

contradicting (2.5).

COROLLARY 2.4. *Problem (2.4) has no radial solutions if* $p \geq n/(n-2)$.

COROLLARY 2.5. *The equation* $\Delta u + u^p = 0$ *in* $\mathbf{R}^n \backslash \{0\} (n \geq 3), (p > 1)$ *has no positive radial solutions behaving like* r^{2-n} *near* $r = 0$ *and* $r = \infty$.

REMARK 2.4. For classical solution in an exterior domain, Noussair and Swanson

[NoS1] have obtained the sharperness of

$$\int_c^\infty r^{n-1-p(n-2)} A(r) dr < \infty.$$

for some $c > 0$.

REMARK 2.5. Theorem 4.1 by Kawano [KA] is for entire solutions in $\mathbf{R}^n$. However, the arguments in his proof are valid when the coefficient of V^p is continuous on $(0, \infty)$.

§3. SINGULAR SOLUTIONS FOR HENON'S TYPE EQUATIONS IN $B_{r_0} \backslash \{0\}$.

Our objective in this section is to prove Theorems A and B in the Introduction. To prove Theorem A we start with a lemma.

Consider

$$(3.1) \begin{cases} v'' + g(t, v) = 0, & t > t_0 (> 0) \\ v(t_0) = 0, \end{cases}$$

where g is continuous on $[t_0, \infty) \times [0, \infty)$.

The following lemma is an improved version of Theorem 3.1 of Noussair and Swanson [NoS2].

Lemma 3.1. *Let* $g(t, s) \geq 0$ *for* $s \geq 0$ *and* $t \geq t_0$. *Assume there exists a function* $\bar{g}(t, s)$, *nondecreasing in* $s \geq 0$, *for each* $t \geq t_0$, *such that*

$$g(t, s) \leq s \bar{g}(t, s) \quad \text{for } t \geq t_0, s \geq 0.$$

Moreover, suppose there exists $\delta > 0$ *with*

$$\int_{t_0}^{\infty} t\bar{g}(t,\delta t)dt < 1.$$

Then for each $\alpha \in (0,\delta]$ *there exists a solution* v *of (3.1) such that* $v'(t_0) = \alpha$ *and*

$$\alpha(1 - \int_{t_0}^{\infty} t\bar{g}(t,\delta t)dt)(t - t_0) \le v(t) \le \alpha(t - t_0),\ t \ge t_0.$$

Proof. For each $\alpha \in (0,\delta]$,

$$\begin{cases} v'' + g(t,v) = 0 \\ v(t_0) = 0, v'(t_0) = \alpha \end{cases}$$

has a local solution v. Let $[t_0, t_\alpha)$ be the maximal interval on which v is positive. We assert that $t_\alpha = +\infty$. Suppose $t_\alpha < \infty$. Since $v(t) \le v'(t_0)(t - t_0) \le \alpha t$ on $[t_0, t_\alpha)$, then for $t_0 < t < t_\alpha$ we have

$$\begin{aligned} \alpha = v'(t_0) = v'(t) + \int_{t_0}^{t} g(s, v(s))ds \\ \le v'(t) + \alpha \int_{t_0}^{\infty} s\bar{g}(s,\delta s)ds. \end{aligned}$$

Thus

$$v'(t) \ge \alpha(1 - \int_{t_0}^{\infty} t\bar{g}(t,\delta t)dt) > 0, \quad t_0 < t < t_\alpha. \tag{3.2}$$

Therefore $\lim_{t\to t_\alpha^-} v'(t)$ exists and is finite. Furthermore, it is easily seen that $\lim_{t\to t_\alpha^-} v(t)$ exists and is finite, proving continuability of v through t_α. We must have $v(t_\alpha) = 0$, and then $v'(t_\alpha) \le 0$, contradicting (3.2).

REMARK 3.1. The existence of δ in Lemma 3.1 is guaranteed if $\int_{t_0}^{\infty} t\bar{g}(t,0)dt < 1$.

Let B_{r_0} be the ball centered at 0 with radius r_o in $\mathbf{R}^n (n \ge 3)$.

We consider

$$(3.3)\begin{cases} \Delta u + f(|x|, u) = 0 & \text{in } \ B_{r_0}\backslash\{0\} \\ u > 0 & \text{in } \ B_{r_0}\backslash\{0\} \\ u = 0 & \text{on } \ \partial B_{r_0}, \end{cases}$$

where f is continuous on $(0, r_0] \times [0, \infty)$.

Proof of Theorem A.

Let

$$w(y) = |y|^{2-n}u(x), \qquad x = \frac{y}{|y|^2}.$$

Then (3.3) is transformed into

$$(3.4)\begin{cases} \Delta w + |y|^{-2-n}f(\frac{1}{|y|}, |y|^{n-2}w) = 0, & |y| > 1/r_0 \\ w > 0 & , \ \ |y| > 1/r_0 \\ w = 0 & , \ \ |y| = 1/r_0. \end{cases}$$

Thus, we consider

$$(3.5)\begin{cases} w''(\tau) + \frac{n-1}{\tau}w'(\tau) + \tau^{-2-n}f\left(\frac{1}{\tau}, \tau^{n-2}w(\tau)\right) = 0, & \tau > 1/r_0 \\ w > 0 & , \ \ \tau > 1/r_0, \\ w(1/r_0) = 0 \end{cases}$$

or equivalently,

$$(3.5)'\begin{cases} v'' + g(t, v) = 0, & t > t_0 \\ v(t) > 0 & , \ \ t > t_0 \\ v(t_0) = 0, \end{cases}$$

where $t = (n-2)\tau^{n-2}$, $v(t) = w(\tau)t$, $\quad t_0 = (n-2)r_0^{2-n}$ and

$$g(t, v) = t^{-3}[\alpha(t)]^{n-4}f\left(\frac{1}{\alpha(t)}, \ [\alpha(t)]^{n-2}v/t\right),$$

with $\alpha(t) = [t/(n-2)]^{1/n-2}$.

Let $\bar{g}(t, s) = t^{-3}[\alpha(t)]^{n-4}\bar{f}\left(\frac{1}{\alpha(t)}, \ \frac{s}{n-2}\right)\left(\frac{1}{n-2}\right)$. We have

$$\begin{aligned} g(t, s) &= t^{-3}[\alpha(t)]^{n-4}f\left(\frac{1}{\alpha(t)}, \frac{s}{n-2}\right) \\ &\le \frac{t^{-3}[\alpha(t)]^{n-4}s}{n-2}\bar{f}\left(\frac{1}{\alpha(t)}, \frac{s}{n-2}\right) \\ &= s\bar{g}(s, t), \end{aligned}$$

and

$$\int_{t_0}^{\infty} t\bar{g}(t,\delta t)dt = (n-2)^{-1}\int_0^{r_0} r\bar{f}(r,\delta r^{2-n})dr < 1.$$

Thus the assumptions of Lemma 3.1 are satisfied and the result follows.

Corollary 3.2. *Let* f *be continuous on* $(0,r_0]\times[0,\infty)$, *with* $0\le f(r,u)\le B(r)u+A(r)u^p, p>1$. *Then the conclusion of Theorem A holds provided*

$$\int_0^{r_0} rB(r)dr < n-2$$

and

$$\int_0^{r_0} r^{1+(p-1)(2-n)}A(r)dr < \infty. \tag{3.6}$$

Proof Let $\bar{f}(r,u) = B(r) + A(r)u^{p-1}$.

Corollary 3.3 *Let* A *be continuous on* $(0,r_0]$ *with* $0\le A(r) = O(r^{-\ell})$ *near* $r=0$, $\ell<2$.

Then

$$\begin{cases} \Delta u + A(r)u^p = 0 & in \;\; B_{r_0}\backslash\{0\} \\ u>0 & in \;\; B_{r_0}\backslash\{0\} \\ u=0 & on \;\; \partial B_{r_0}. \\ u\sim r^{2-n} & near \;\;\; r=0 \end{cases}$$

has infinitely many radial solutions provided $1<p<(n-\ell)/(n-2)$.

Corollary 3.4 *For* $1<p<n/(n-2)$, *the Matukuma's equation in* $B_{r_0}\backslash\{0\}$ *has infinitely many positive radial solutions vanishing on* ∂B_{r_0} *and behaving like* r^{2-n} *near* $r=0$.

Proof: Take $\ell=0$ in Corollary 3.3.

REMARK 3.2. Corollary 3.3 includes Hénon's model [H] where $A(r)=|x|^{-\ell}$ and u represents the gravitational potential of a stationary rotating stellar system.

REMARK 3.3. It is interesting to note that the integrand in the integrability condition (3.6) is the same as in (2.2) for Matukuma's type equations.

REMARK 3.4. It is easy to show that the solutions obtained in Theorem A have finite total mass (in $B_{r_0}\backslash\{0\}$).

REMARK 3.5. A more detailed asymptotic expansion near the origin can also be given using the results by Li and Ni [LN], and Li [L1] applied to (3.4).

Next we show that the integrability condition in Theorem A is sharp in the following sense.

THEOREM 3.5 *Let A be C^1 on $[0, r_0]$ and nonnegative. If*

$$\int_0^{r_0} r^{1+(p-1)(2-n)} A(r)dr = \infty,$$

then

$$(3.7)\begin{cases} \Delta u + A(r)u^p = 0 & in \;\; B_{r_0}\backslash\{0\}, \\ u > 0 & in \;\; B_{r_0}\backslash\{0\}, \\ u \sim r^{2-n} & near \;\; r = 0 \end{cases}$$

has no radial solution.

Proof: This follows easily from Lemma 1.1 in Ni and Sacks [NS1]. In fact,

$$\infty > \int_{B_{r_0}\backslash\{0\}} A(|x|)u^p(x)dx = \int_0^{r_0} r^{n-1}A(r)u^p(r)dr,$$

contradicting the integrability assumption.

Corollary 3.6. *If A is C^1 on $[0, r_0]$, nonnegative, and $A(r) \geq cr^{-\ell}$ near $r = 0$ for some positive constant c. Then (3.7) has no radial solution provided*

$$p \geq (n - \ell)/(n - 2).$$

This corollary shows the sharperness of the range of p ($p < (n - \ell)/(n - 2), \ell \leq -1$) in Hénon's model (Corollary 3.4 above). It also applies to the Matukuma's equation ($\ell = 0$) in the punctured ball.

If A is not necessarily C^1, we may still have nonexistence results; e.g., Theorem B.

Proof of Theorem B.

It suffices to show that Problem $(3.5)'$ with $g(t,v) = a(t)v^p$ has no solution. Here,

$$a(t) \equiv t^{-3-p}[\alpha(t)]^{n-4}A(1/\alpha(t))[\alpha(t)]^{p(n-2)},$$

$t = (n-2)r^{n-2}, r = \alpha(t)$.

Applying Corollary 10 of Coffman and Wong [CW], any solution of $(3.5)'$ is oscillatory if

$$t^{(p+3)/2}a(t)$$

is nondecreasing. A simple calculation shows that

$$t^{(p+3)/2}a(t) = cr^{-\eta}A(1/r),$$

where c is a positive constant and $\eta = \frac{n+2-p(n-2)}{2}$

Corollary 3.7. *Let $p > 1$ and $\ell < 2$. Then*

$$(3.8)\begin{cases} \Delta u + |x|^{-\ell}u^p = 0 & in \;\; B_{r_0}\backslash\{0\}, \\ u > 0 & in \;\; B_{r_0}\backslash\{0\}, \\ u = 0 & on \;\; \partial B_{r_0}, \end{cases}$$

has no radial solution provided

$$p \geq (n+2-2\ell)/(n-2).$$

Note that in Corollary 3.7, *no restriction is imposed at the origin.*

The nonexistence result of singular solutions for (3.8) with $\ell \geq 0$ and $p \geq (n+2-2\ell)/(n-2)$ is due to Ni and Sacks [NS1]. If $0 \leq \ell < 2$ and

$$(n-\ell)/(n-2) < p < (n+2-2\ell)/(n-2)$$

it follows from Theorem 3.3 by Gidas and Spruck [GS] that

$$(3.8)' \begin{cases} \Delta u + |x|^{-\ell} u^p = 0 & \text{in } B_{r_0}\backslash\{0\} \\ u > 0 & \text{in } B_{r_0}\backslash\{0\} \\ u \sim r^{2-n} & \text{near } r = 0 \end{cases}$$

has no solution.

If $p = (n-\ell)/(n-2)$ and $-2 < \ell < 2$, the nonexistence result for $(3.8)'$ follows from the work by Aviles [A]. Finally, if $\ell \geq 2$ and $p > 1$, it is known [N1] that $\Delta u + A(x)u^p = 0$ in $B_{r_0}\backslash\{0\}$, where $A \sim |x|^{-\ell}$, does not possess any solutions. We do not know of any results for $(3.8)'$ covering the case $-1 \leq \ell < 0$ and

$$(n-\ell)/(n-2) < p < (n+2-2\ell)/(n-2).$$

The next result closes the gap.

THEOREM 3.8. *Let A be continuous and nonnegative on $(0, r_o]$ with $A(r) \sim r^{-\ell}$, $-1 \leq \ell < 0$ and*

$$(n-\ell)/(n-2) < p < (n+2-2\ell)/(n-2).$$

Then

$$\begin{cases} \Delta u + A(r)u^p = 0 & \textit{in } B_{r_0}\backslash\{0\} \\ u > 0 & \textit{in } B_{r_0}\backslash\{0\} \\ u \sim r^{2-n} & \textit{near } r = 0 \end{cases}$$

has no solution.

Proof. Suppose there is a solution. Let

$$\bar{u}(r) = \int\limits_{|x|=r} u(x) ds_x.$$

$\bar{u}$ is radial and

$$\bar{u}'' + \frac{n-1}{r}\bar{u}' + A(r)\bar{u}^p(r) \leq 0$$

with $\bar{u}(r) \sim r^{2-n}$ near 0 or $\bar{u}(r) \geq C_1 r^{2-n}, A(r) \geq C_2 r^{-\ell}$.

For a fixed $r_1 > 0$, and any $0 < r < r_1$, we have

$$(r^{n-1}\bar{u}')' + r^{n-1}A(r)\bar{u}^p(r) \leq 0 \qquad \text{or}$$

$$\int_r^{r_1} r_1^{n-1}\bar{u}'(r_1) - r^{n-1}\bar{u}'(r) + \int_r^{r_1} s^{n-1}A(s)\bar{u}^p(s)ds \leq 0$$

$$\frac{r_1^{n-1}\bar{u}'(r_1)}{r^{n-1}} + \frac{1}{r^{n-1}}\int_r^{r_1} s^{n-1}A(s)\bar{u}^p(s)ds$$

$$\leq \bar{u}'(r)\int_r^{r_1} \frac{r_1^{n-1}\bar{u}'(r_1)}{t^{n-1}}dt + \int_r^{r_1} \frac{dt}{t^{n-1}}\int_t^{r_1} s^{n-1}A(s)\bar{u}^p(s)ds \leq \bar{u}'(r_1) - \bar{u}(r)$$

$$\bar{u}(r) \leq \bar{u}(r_1) - \int_r^{r_1} \frac{r_1^{n-1}\bar{u}'(r_1)}{t^{n-1}}dt -$$

$$-\int_r^{r_1} \frac{dt}{t^{n-1}}\int_t^{r_1} s^{n-1}A(s)\bar{u}^p(s)ds$$

$$= \bar{u}(r_1) + \frac{r_1\bar{u}'(r_1)}{n-2} - \frac{r_1^{n-1}\bar{u}'(r_1)}{(n-2)r^{n-2}} -$$

$$-c_2c_1^p\int_r^{r_1} \frac{dt}{t^{n-1}}\int_t^{r_1} s^{n-1-\ell-p(n-2)}ds$$

where

$$\ell - 3 < n - 1 - \ell - p(n-2) < -1$$

and hence

$$\bar{u}(r) \leq \bar{u}(r_1) + \frac{r_1 \bar{u}'(r_1)}{n-2} - \frac{r_1^{n-1}\bar{u}'(r_1)}{(n-2)r^{n-2}} + \frac{c_1^p c_2}{p(n-2)-\ell-n} \times$$

$$\int_r^{r_1} \frac{dt}{t^{n-1}} \left[s^{n-1-\ell-p(n-2)} \Big|_t^{r_1} \right]$$

$$\begin{aligned}\bar{u}(r) \leq & \bar{u}(r_1) + \frac{r_1 \bar{u}'(r_1)}{n-2} - \frac{r_1^{n-1}\bar{u}'(r_1)}{(n-2)r^{n-2}} \\ & - \frac{c_1^p c_2}{p(n-2)+\ell-n} - \frac{1}{n-2} r_1^{2-\ell-p(n-2)} \\ & + \frac{c_1^p c_2}{p(n-2)+\ell-n} - \frac{1}{n-2} r_1^{n-\ell-p(n-2)} \cdot \frac{1}{r^{n-2}} \\ & + \frac{c_1^p c_2 r_1^{2-\ell-p(n-2)}}{[p(n-2)+\ell-n][p(n-2)+\ell-2]} \\ & - \frac{c_1^p c_2}{[p(n-2)+\ell-n][p(n-2)+\ell-2]} \frac{1}{r^{p(n-2)+\ell-2}}\end{aligned}$$

and since $n - \ell > p(n-2) + \ell - 2 > n - 2$, we have $\bar{u}(r) < 0$ for small $r > 0$, which is a contradiction.

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