

# Separation property of solutions for a semilinear elliptic equation

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## Abstract

In this paper, we study the following elliptic problem

$$\begin{cases} \Delta u + K(x)u^p = 0 & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \end{cases} \quad (*)$$

where  $K(x)$  is a given function in  $C^\alpha(\mathbb{R}^n \setminus 0)$  for some fixed  $\alpha \in (0, 1)$ ,  $p > 1$  is a constant. Some existence, monotonicity and asymptotic expansion at infinity of solutions of (\*) are discussed.

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## 1 Introduction

In this paper, we study the existence, monotonicity and asymptotic expansion at infinity of positive solutions for the semilinear elliptic equation

$$\Delta u + K(|x|)u^p = 0, \quad (1.1)$$

where  $p > 1$ ,  $x \in \mathbb{R}^n$ ,  $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$  is the  $n$ -dimensional Laplacian,  $K(x) \in C^\alpha(\mathbb{R}^n \setminus 0)$  for some fixed  $\alpha \in (0, 1)$  is a given function.

Equation (1.1) arises from both physics and mathematics. For  $K(x) \equiv |x|^l$ , equation (1.1) is known as Lane-Emden equation, sometimes it is also referred to as the Emden-Fowler equation in astrophysics, where  $u$  represents the density of a single star. In geometry when  $p = \frac{n+2}{n-2}$ ,  $n \geq 3$ , equation (1.1) is called the conformal scalar curvature equation in  $\mathbb{R}^n$ . Let  $g$  be the usual metric in  $\mathbb{R}^n$ , the problem of finding a metric  $g_1$  which is conformal to  $g$  (i.e.  $g_1 = u^{\frac{4}{n-2}}g$ , for some positive function  $u$  with scalar curvature  $\tilde{K}$ ) is equivalent to finding a positive solution of (1.1) with  $K = \frac{n-2}{4(n-1)}\tilde{K}$ . For a detailed overview on (1.1), we refer readers to the survey paper [N2] by Ni.

Because  $K \in C^\alpha(\mathbb{R}^n \setminus 0)$  for some fixed  $\alpha \in (0, 1)$ , bounded solutions of (1.1) are classical on  $0 < |x| < \infty$ . However, at  $x = 0$ , where  $K$  is “bad”, usually we can not expect the solutions to be differentiable, or even continuous owing to the singularity of  $K$  at  $x = 0$ . Let  $u$  be a solution of (1.1), the singular point  $x = 0$  of (1.1) is called a *removable singular point* of  $u(x)$  if  $u(0) \equiv \lim_{x \rightarrow 0} u(x)$  exists, otherwise  $x = 0$  is called a *nonremovable singular point*. It is showed by Ni and Yotsutani ([NY]) that when  $x = 0$  is a removable singular point of a regular solution, the existence of the derivatives of the solution depends on the “blow-up” rate of  $K$  at  $x = 0$  ([NY] Proposition 4.4)

**Definition 1.1** . Let  $u \in C^2(\mathbb{R}^n \setminus 0)$  be a solution of (1.1). If  $x = 0$  is a removable singular point of  $u$ , then  $u$  is said to be a regular solution of (1.1). If  $x = 0$  is a nonremovable singular point of  $u$ , then  $u$  is said to be an irregular solution of (1.1).

For the physical reasons and because of the results on the symmetry of positive solutions (see, for example, [CGS], [CL1,2], [GNN 1,2], [G1,2], [Li], [L2], [LN2,3,4,5], [YY1,2] and references therein) we consider the positive radial solutions of (1.1), with  $r = |x|$  and equation (1.1) reduces to

$$u'' + \frac{n-1}{r}u' + K(r)u^p = 0, \quad r > 0. \quad (1.2)$$

For the same reasons, the regular solutions are of particular interest. This leads us to consider the initial value problem

$$\begin{cases} u'' + \frac{n-1}{r}u' + K(r)u^p = 0 & r > 0, \\ u(0) = \alpha > 0. \end{cases} \quad (1.3)$$

In this paper, we use notation  $u_\alpha = u(r; \alpha)$  to denote the solution of (1.3).

Equation (1.1) was studied by many mathematicians. It is showed ([N1] and [Lin]) that if  $K(r) \leq -Cr^{(n-2)(p-1)-2}$  at infinity for some constant  $C > 0$ , then (1.1) possesses no positive solutions. In case when  $|K| \leq Cr^{(n-2)(p-1)-2-\delta}$  at infinity for some positive constants  $C$  and  $\delta$ , the existence and asymptotics at infinity of positive solutions are studied by many authors, here we only mention the results of, for example, W.-M. Ni and S Yosutani [NY] and Y. Li [L1]. In this so-called fast decay case, Ni showed that (1.1) possesses infinitely many positive solutions which are bounded from below by positive constants (see [N1] and [Lin]). Li and Ni ([LN1]) showed that for positive solutions of (1.1) the limit  $u_\infty = \lim_{x \rightarrow \infty} u(x)$  always exists; furthermore, if  $u_\infty = 0$ , then for any  $\varepsilon > 0$ ,

$$u(x) \leq \begin{cases} C|x|^{2-n} & \text{if } p > \frac{n+l}{n-2}, \\ C_\varepsilon|x|^{\frac{(1-\varepsilon)(l+2)}{1-p}} & \text{if } p \leq \frac{n+l}{n-2}, \end{cases}$$

where  $C_\varepsilon$  is a constant depending on  $\varepsilon$  and  $l$  is the decay rate of  $K$  (please refer to (K.1-4)); and if  $u_\infty > 0$ , then

$$|u - u_\infty| \leq \begin{cases} C|x|^{2-n} & \text{if } l < -n, \\ C|x|^{2-n} \log |x| & \text{if } l = -n, \\ C|x|^{2+l} & \text{if } -n < l < -2, \end{cases}$$

at  $\infty$ . Li refined these results and gave the limit  $u_\infty$  explicitly in terms of  $n, p, K$  (see [L1] or Theorem B in this section.)

In this paper, we will focus on the slow decay case, i.e.,  $K(r) \geq Cr^l$ , for some  $l > -2$  and  $r$  large.

First, let us introduce a collection of hypotheses on  $K$ .

(K.1).  $K(r) > 0$  in  $r > 0$  and  $\lim_{r \rightarrow \infty} r^{-l}K(r) = k_\infty > 0$ , where  $l > -2$ ,

(K.1').  $K(r) > 0$  in  $r > 0$  and  $\lim_{r \rightarrow 0^+} r^{-l}K(r) = k_0 > 0$ , where  $l > -2$ ,

(K.2).  $K(r)$  is differentiable and  $[\frac{d}{dr}(r^{-l}K(r))]^+ \in L^1$ ,  $r$  near  $\infty$ ,

(K.3).  $K(r)$  is differentiable and  $[\frac{d}{dr}(r^{-l}K(r))]^- \in L^1$ ,  $r$  near  $\infty$ ,

(K.4).  $K(r)$  is differentiable and  $\frac{d}{dr}(r^{-l}K(r)) \leq 0$ ,  $r > 0$ .

Also we introduce the following notations which will be used throughout this paper:

$$m \equiv \frac{l+2}{p-1}, \quad b_0 \equiv n-2-2m$$

$$L \equiv [m(n-2-m)]^{\frac{1}{p-1}}, \quad c_0 \equiv (p-1)L^{p-1} \quad (1.4)$$

It is easy to see that in the slow decay case  $l > -2$ , when  $p > \frac{n+2l+2}{n-2}$ , we have  $m > 0$  and  $b_0 > 0$ .

The existence of the solution of (1.3), which is obtained by Ni and Yasutani, is stated as follows.

**Theorem A.** (Theorem 6 in [NY]) *Suppose that (K.1') and (K.4) hold, and  $m \leq (n-2)/2$ , then for every  $\alpha > 0$ , (1.3) has exactly one solution  $u(r) > 0$ , and  $u(0) = \alpha$ . Moreover, if  $\frac{d}{dr}(r^{-l}K(r)) \not\equiv 0$  on  $(0, +\infty)$ , or  $m < (n-2)/2$ , then*

$$\int_0^\infty K(r)u^p r^{n-1} dr = \infty.$$

**Remark 1.1** . In fact, condition (K.1') is stronger than what is assumed in Theorem 6 in [NY].

In the case when  $m > (n - 2)/2$ , if  $r^{-l}K(r)$  has a positive limit at  $r = 0^+$ , then there exists  $\alpha_1 > 0$  such that for every  $\alpha \geq \alpha_1$ , equation (1.3) has no entire positive solution with initial value  $\alpha$ . This is the result of Theorem 2 in [NY]. Under such sense  $m = \frac{n-2}{2}$  is a critical index to the problem (1.3) . The existence of positive solutions is also established in [DN] and [LN2]. Under various assumptions on  $K$ , uniqueness of positive solutions is obtained in [KL] and [YY1].

The following theorem is obtained by Li, giving an accurate description on the asymptotic behavior of positive solutions of (1.1).

**Theorem B.** ( Theorem 1, [L1]) *Let  $u$  be a positive radial solution of (1.1). Assume that  $K$  satisfies*

- (i) (K.1) and (K.2), if  $0 < m < (n - 2)/2$ , or
- (ii) (K.1) and (K.3), if  $(n - 2)/2 < m < n - 2$ .

Then

$$\lim_{r \rightarrow \infty} r^m u(r) = u_\infty \equiv \begin{cases} \frac{L}{k_\infty^{\frac{1}{p-1}}}, & \text{or} \\ 0. \end{cases}$$

Furthermore, if  $u_\infty = 0$ , then

$$\lim_{r \rightarrow \infty} r^{n-2} u(r)$$

exists and is finite and positive.

**Remark 1.2** . When  $l = -2$ , a result similar to Theorem B holds (See [LN2] and [L1,2].)

A natural and interesting question concerning equation (1.3) is: do two solutions with different initial values intersect each other, or, in other words, do the solutions of (1.3) depend on  $\alpha$  monotonically (i.e. have separation property)? It is known that the monotone property of the solutions of (1.3) has important applications, such as stability, etc.

It is showed by Wang([W]), Ni and Yosutani ([NY]) that for small  $p$ , any two positive solutions intersect each other. Wang also showed that for large  $p$ , the solutions of (1.3) possess monotone property for a special class of  $K$ , and gave explicitly the lower bound of the  $p$  value.

In case of  $K(r) = r^l$ ,  $l > -2$ , the following

$$U_s(r) = Lr^{-m} \quad (1.5)$$

is a singular solution of equation (1.3) with  $K(r) = r^l$ . To state Wang's result, we define constant  $p_c$  by

$$p_c = \begin{cases} \frac{(n-2)^2 - 2(l+2)(n+l) + 2(l+2)\sqrt{(n+l)^2 - (n-2)^2}}{(n-2)(n-10-4l)} & n > 10 + 4l, \\ \infty & 3 \leq n \leq 10 + 4l, \end{cases}$$

particularly, when  $l = 0$  we have,

$$p_c = \begin{cases} \frac{(n-2)^2 - 4n + 4\sqrt{n^2 - (n-2)^2}}{(n-2)(n-10)} & n > 10 \\ \infty & 3 \leq n \leq 10. \end{cases}$$

We have

**Theorem C.** ([W] Proposition 3.7.(iii), (iv)) *Let  $u_\alpha(r)$  be the solution of (1.3) with  $K(r) = r^l$ . Then we have*

(i) *when  $(n+2+2l)/(n-2) < p < p_c$ , the graph of  $u_\alpha(r)$  oscillates around that of  $U_s(r)$  for every  $\alpha > 0$ ,*

(ii) *when  $p \geq p_c$ , the graph of  $u_\alpha$  does not intersect that of  $U_s$  (i.e.,  $u(r) < U_s(r)$ ) for every  $\alpha > 0$ . Furthermore,  $u_\alpha(r)$  is increasing with respect to  $\alpha$ .*

**Remark 1.3** . *If  $K(r) = r^l$ ,  $p > \frac{n+2+2l}{n-2}$ ,  $l > -2$ , then the nontrivial regular solution  $u_\alpha$  of (1.4) can be expressed in terms of  $u_1(r)$ , the solution of (1.3) with initial value  $\alpha = 1$ , and  $u_\alpha = \alpha u_1(\alpha^{\frac{1}{m}}r)$ , and  $U_s(r)$  is the only singular solution of (1.3) (see [GS1,2], [W].)*

For large  $p$ , Theorem C is extended to a more general class of  $K$  by Gui [G1,2]. We cite Gui's result in the following version.

**Theorem D.** ([G1] Lemma 3.1) *Assume  $K$  satisfies*

$$\begin{cases} a_1 r^{l_1} \leq K(r) \leq b_1 r^{l_1} & \text{for } r \leq 1, \\ a_2 r^{l_2} \leq K(r) \leq b_2 r^{l_2}, & \text{for } r \geq 1 \end{cases}$$

where  $l_1, l_2 > -2$ ,  $a_1, a_2, b_1$  and  $b_2$  are positive constants, and  $0 < a_1 < a_2, 0 < b_1 < b_2$ .

Define  $F$  by

$$F = \frac{4}{(n-2)^2} \max \left\{ \frac{(n+l_1)(l_1+2)}{a_1}, \frac{(l_2+2)[(n+l_2)a_1 + (n+l_1)a_2]}{a_1 a_2} \right\} \max\{b_1, b_2\}.$$

Let  $u_\alpha(r)$  and  $u_\beta(r)$  be two regular solutions of equation (1.3) with initial values  $0 < u_\alpha(0) < u_\beta(0)$ . If  $p \geq \frac{1}{1-F}$ , for  $F < 1$ , then we have  $u_\beta(r) > u_\alpha(r)$  for  $r \in [0, \infty)$ .

**Remark 1.4** . When  $p \geq 1 + F$ , Theorem 2.6 in [G1] guarantees that the solutions in Theorem C are positive.

Our main results obtained on equation (1.1) are as follows.

First, we study the monotonicity of solutions of (1.3) with respect to the initial data  $\alpha$  and get a sharp estimate  $p_c$  on the exponent  $p$  under more general condition imposed on  $K$ . More exactly, we have

**Theorem 1.** Suppose that  $K(r)$  satisfies (K.1), (K.1') and (K.4). Let  $u_\alpha(r) = u_\alpha(r, \alpha)$  and  $u_\beta(r) = u_\beta(r, \beta)$  be two positive solutions of equation (1.3) with  $u_\alpha(0) = \alpha$ ,  $u_\beta(0) = \beta$ , and  $0 < \alpha < \beta$ . Then

- (i) when  $p > p_c$ ,  $u_\alpha(r)$  and  $u_\beta(r)$  can not intersect each other, i.e.,  $u_\alpha(r) < u_\beta(r)$ .
- (ii) when  $(n+2+2l)/(n-2) < p < p_c$ ,  $u_\alpha(r)$  and  $u_\beta(r)$  will intersect infinity many times,

Secondly, we study the singular solutions of equation (1.3), which blow up at  $r = 0$ , and we give a general uniqueness theorem.

**Theorem 2.** Suppose  $K(r)$  satisfies (K.1') and (K.2),  $p > (n+2+2l)/(n-2)$ . Then equation (1.3) has one and only one singular solution  $U(r)$ , which satisfies

$$\lim_{r \rightarrow 0^+} r^m U(r) = \frac{L}{k_0^{\frac{1}{p-1}}}, \quad \lim_{r \rightarrow 0^+} r^{m+1} U'(r) = -m \frac{L}{k_0^{\frac{1}{p-1}}}.$$

Furthermore, if  $p > p_c$ ,  $K(r)$  satisfies (K.1), (K.1'), and (K.4), then for any regular solution  $u(r)$ , the following holds

$$u(r) < U(r) \leq \frac{L}{(r^{-l}K(r))^{\frac{1}{p-1}}r^m}.$$

This paper is organized as follows: We first give an estimate on solutions of equation (1.3) in Section 2. The proof of Theorem 1 then is given in Section 3. we study singular solutions and prove Theorem 2 in Section 4. Finally, the asymptotic expansions of solutions of (1.3) are given in Section 5

## 2 An Estimate

In this section, we will give an estimate on the solutions of equation (1.3) . This is crucial to the proof of Theorem 1.

Without any particular statement, all solutions appearing in this and the following sections are regular ones. First, let us introduce the following transformation, which will be used frequently in this and later sections.

**Lemma 2.1** . Suppose that  $u$  is a positive solution of (1.3). Let  $r = e^t$ ,  $t \in (-\infty, +\infty)$  and  $v(t) = r^q u(r)$ , then  $v$  satisfies

$$v'' + (n - 2 - 2q)v' - q(n - 2 - q)v + K(e^t)e^{(q+2-pq)t}v^p = 0. \quad (2.1)$$

This Lemma can be proved by straightforward calculations, we omit it here.

Following the approach by Wang ([W]), we have.

**Lemma 2.2** . Suppose that  $K(r)$  satisfies (K.1), (K.1') and (K.4). Let  $u(r)$  be the positive solution of (1.3). If  $p \geq p_c$ , then  $r^m u(r)$  is strictly increasing in  $r$  and

$$(r^{-l}K(r))(r^m u(r))^{p-1} < L^{p-1}. \quad (2.2)$$

**Proof:** Let  $q = m$  in Lemma 2.1, then we have that

$$v'' + b_0v' - L^{p-1}v + k(t)v^p = 0, \quad (2.3)$$

here  $k(t) = e^{-lt}K(e^t)$ ,  $v = e^{mt}u(e^t)$  and  $m$ ,  $b_0$ , and  $L$  are as in (1.5). We need only to show that  $k(t)v^{p-1} < L^{p-1}$ . By (K.1) (K.1') and (K.4),  $\lim_{t \rightarrow \infty} k(t) = k_\infty$ ,  $\lim_{t \rightarrow -\infty} k(t) = k_0 > 0$  and  $k'(t) \leq 0$  for  $t \in \mathbb{R}$ . Since  $v(t) > 0$  for  $t \in \mathbb{R}$ , and  $\lim_{t \rightarrow -\infty} v(t) = 0$ , we have  $k(t)v^{p-1} < L^{p-1}$  at the neighborhood of  $t = -\infty$ .

On the contrary, suppose that there exists  $t \in \mathbb{R}$ , such that  $k(t)v^{p-1} \geq L^{p-1}$ . Let

$$T = \min\{t \in \mathbb{R} \mid k(t)v^{p-1} \geq L^{p-1}\}. \quad (2.4)$$

then  $T > -\infty$ ,  $k(t)v^{p-1} < L^{p-1}$  for  $t < T$  and  $k(T)v^{p-1}(T) = L^{p-1}$ . From (2.3) we have that

$$v'' + b_0v' > 0 \quad (2.5)$$

for all  $t < T$ . This implies  $e^{b_0t}v'$  is strictly increasing on  $(-\infty, T)$ . By Proposition 4.1.(b) in [NY] and the facts that both  $r^{-l}K(r)$  and  $u(r)$  are bounded, from equation (1.3) we have that

$$\begin{aligned} u'(r) &= - \int_0^r \left(\frac{s}{r}\right)^{n-1} K(s)u^p(s)ds \\ &= O\left(r^{l+1}\right) \end{aligned}$$

at  $r = 0^+$ . Hence  $v'(t) = mr^m u(r) + r^{m+1}u'(r)$ ,  $r = e^t$ , goes to zero as  $t \rightarrow -\infty$ . Since  $b_0 > 0$ , we have that  $e^{b_0t}v'(t) \rightarrow 0$  as  $t \rightarrow -\infty$ , and  $v'(t) > 0$  for  $t \in (-\infty, T)$ . Let  $q(v) = v'(t) > 0$  for  $(0, \frac{L}{k(T)^{\frac{1}{p-1}}})$ , then  $q(v) > 0$ ,  $q(v) \rightarrow 0^+$  as  $v \rightarrow 0^+$ , and satisfies

$$\frac{dq}{dv} = -b_0 + \frac{L^{p-1}v - k(t)v^p}{q}. \quad (2.6)$$

Therefore in the  $q - v$  plane the line  $q = \mu(\frac{L}{(k(T))^{\frac{1}{p-1}}} - v)$  must intersect the graph of  $q(v)$  for every  $\mu > 0$ . Let  $(v_\mu, q(v_\mu))$  be the intersection with the smallest  $v$ -coordinate for each  $\mu > 0$ , then we have  $(dq/dv) \geq -\mu$  at  $(v_\mu, q(v_\mu))$ ; moreover, the following holds at  $(v_\mu, q(v_\mu))$

$$\frac{dq}{dv}(v_\mu) = -b_0 + \frac{L^{p-1}v_\mu - k(t)v_\mu^{p-1}}{\mu \left( \frac{L}{k(T)^{\frac{1}{p-1}}} - v_\mu \right)}. \quad (2.7)$$

Since  $k'(t) \leq 0$ , it follows that  $k(t) \geq k(T)$  for  $t \leq T$ , and by mean value theorem, there exists  $\tilde{v}_\mu \in (v_\mu, L^{p-1}/k(T))$  such that

$$\begin{aligned} -\mu &\leq -b_0 + \frac{k(T)v_\mu \left( \frac{L^{p-1}}{k(T)} - v_\mu^{p-1} \right)}{\mu \left( \frac{L}{k(T)^{\frac{1}{p-1}}} - v_\mu \right)} \\ &= -b_0 + \frac{(p-1)k(T)v_\mu \tilde{v}_\mu^{p-2}}{\mu} \\ &< -b_0 + \frac{(p-1)L^{p-1}}{\mu}. \end{aligned}$$

with  $\tilde{v}_\mu \in (v_\mu, L^{p-1}/k(T))$ . So we have

$$\mu^2 - b_0\mu + c_0 > 0 \quad (2.8)$$

holding for all  $\mu > 0$ , so the determinant of the quadratic form (2.8) must be negative, i.e.,  $b_0^2 - 4(p-1)L^{p-1} < 0$ . By direct calculations, (2.8) holds if and only if  $p < p_c$ . The contradiction shows  $k(t)v^{p-1} < L^{p-1}$  for all  $t \in \mathbb{R}^1$ , hence (2.2) holds. Consequently, (2.5) holds for all  $t \in \mathbb{R}$ . Multiplying (2.5) by  $e^{b_0 t}$  and integrating over  $(-\infty, t)$ , we get  $v'(t) > 0$  for  $t \in \mathbb{R}$ , hence  $r^m u(r)$  is strictly increasing.  $\square$

**Remark 2.1** . *By Theorem B we know that the limit  $\lim_{r \rightarrow \infty} r^m u(r)$  exists and is either  $\frac{L}{k_\infty^{\frac{1}{p-1}}}$  or 0. Since  $r^m u(r)$  is strictly increasing by Lemma 2.2, it follows that the limit is  $\frac{L}{k_\infty^{\frac{1}{p-1}}}$ .*

### 3 Proof of Theorem 1

The existence of positive solutions of (1.3) is a consequence of Theorem A([NY]). Theorem 1 asserts that when  $p > p_c$  the solutions of equation (1.2) have "layer" property. We divide

the proof into two parts. We first prove Theorem 1 (i) under special assumptions on initial values of solutions, which is stated as follows

**Theorem 3.1** . Suppose that  $p \geq p_c$  and  $K(r)$  satisfies (K.1), (K.1') and (K.4). Let  $u_\alpha(r)$  and  $u_\beta(r)$  are two positive solutions of (1.3) with initial values  $u_\alpha(0) = \alpha$ ,  $u_\beta(0) = \beta$ ,  $\beta > \alpha > 0$ . Then there exists a constant  $\lambda > 1$ ,  $\lambda \sim \lambda(p, n, l)$ , such that if  $\left(\frac{\beta}{\alpha}\right)^{p-1} < \lambda$ , then  $u_\beta(r) > u_\alpha(r)$ ,  $r > 0$ .

**Proof:** As in Section 2, let  $v_i(t) = r^m u_i(r)$ ,  $r = e^t$ ,  $i = \alpha, \beta$ .

By replacing  $v$  by  $v_\alpha$  and  $v_\beta$  respectively, from (2.3) we have

$$v_\alpha'' + b_0 v_\alpha' + k(t) v_\alpha^p - L^{p-1} v_\alpha = 0 \quad (3.1)$$

and

$$v_\beta'' + b_0 v_\beta' + k(t) v_\beta^p - L^{p-1} v_\beta = 0 \quad (3.2)$$

Let  $w = v_\beta/v_\alpha$ . Subtracting (3.1) from (3.2), then dividing by  $v_\alpha^2$ , we have

$$w'' + \left(b_0 + \frac{2v_\alpha'}{v_\alpha}\right) w' + k(t) v_\alpha^{p-1} (w^p - w) = 0. \quad (3.3)$$

Let  $Q = w - 1$ ,  $f(t) = b_0 + \frac{2v_\alpha'}{v_\alpha}$ , and  $g(t) = k(t) v_\alpha^{p-1} (w^p - w)/(w - 1)$  when  $w \neq 1$ ,  $g(t) = (p-1)k(t)v_\alpha^p$  when  $w = 1$ , from (3.3) we have

$$Q'' + f(t)Q' + g(t)Q = 0. \quad (3.4)$$

It is sufficient to show that  $Q > 0$  for  $t \in \mathbb{R}$ . Suppose that this is not true, i.e. there exists  $t$  such that  $Q(t) \leq 0$ . Define

$$T = \min\{t \in \mathbb{R}^1 \mid Q(t) = 0\}. \quad (3.5)$$

Since  $\lim_{t \rightarrow -\infty} Q(t) = \beta/\alpha - 1 > 0$ , so  $T$  is bounded from below,  $Q(t) > 0$  for  $t < T$ , and  $Q(T) = 0$ . From (3.4) we have

$$Q'' + f(t)Q' < 0 \quad t < T. \quad (3.6)$$

Multiplying (3.6) by  $\exp\left(\int_0^t f(s)ds\right)$  and integrating over  $(\tau, t)$  yields

$$\exp\left(\int_0^t f(s)ds\right) Q'(t) < \exp\left(\int_0^\tau f(s)ds\right) Q'(\tau) \quad \tau < t < T. \quad (3.7)$$

From Lemma 2.2 we know  $v_\alpha' > 0$ , hence  $f(t) > b_0$ . Letting  $\tau$  (take a subsequence if necessary) go to  $-\infty$  in (3.7) we have  $Q'(t) < 0$  for  $t \leq T$ . Thus  $0 < w(t) < w(-\infty) = \frac{\beta}{\alpha}$  for  $t < T$ . By mean value theorem,  $g(t) < (p-1)L^{p-1}\left(\frac{\beta}{\alpha}\right)^{p-1} = c_0\left(\frac{\beta}{\alpha}\right)^{p-1}$  for all  $t < T$ , from (3.4) we have

$$Q'' + f(t)Q' + c_0\left(\frac{\beta}{\alpha}\right)^{p-1} Q > 0 \quad t < T. \quad (3.8)$$

Consider the following second order ordinary equation

$$q''(t) + b_0q'(t) + c_0\left(\frac{\beta}{\alpha}\right)^{p-1} q(t) = 0. \quad (3.9)$$

Since  $p > p_c$ , therefore  $b_0^2 - 4c_0 > 0$ ; so there exists a positive number  $\lambda \sim \lambda(p, n, l)$ ,  $\lambda > 1$ , such that  $b_0^2 - 4c_0(\beta/\alpha)^{p-1} > 0$  when  $1 < (\beta/\alpha) < \lambda$ . Under such conditions the characteristic roots of equation (3.9) are negative and their absolute values are less than  $b_0$ . Let  $q(t)$  be a positive characteristic solution of (3.9), then  $e^{b_0t}(|q| + |q'|) \rightarrow 0$  as  $t \rightarrow -\infty$ . Multiplying (3.8) by  $q(t)$ , (3.9) by  $Q(t)$ , and taking the difference, we have that

$$(Q'q - Qq')' + f(t)Q'q - b_0Qq' > 0 \quad t < T. \quad (3.10)$$

Recalling that  $Q' < 0$ ,  $q > 0$  and  $f(t) > b_0$ , from(3.10) we have

$$(Q'q - Qq')' + b_0(Q'q - Qq') > 0 \quad t < T. \quad (3.11)$$

Multiplying (3.11) by  $e^{b_0t}$  and Integrating over  $(\tau, t)$ , we have

$$e^{b_0\tau}(Q'q - Qq')(\tau) < e^{b_0t}(Q'q - Qq')(t) \quad \tau < t < T. \quad (3.12)$$

Since  $Q(t) \rightarrow 0$ ,  $Q'(t) \rightarrow 0$ , as  $t \rightarrow -\infty$ , letting  $\tau \rightarrow -\infty$  and  $t \rightarrow T$  in (3.12), we have  $Q'(T) \geq 0$ . This is impossible by O.D.E. theory. The contradiction completes our proof.

Q.E.D.

**Proof of Theorem 1(i):** For fixed  $p > p_c$ , if  $\beta > \alpha > 0$  are any two positive numbers,  $u_\alpha(r)$  and  $u_\beta(r)$  are two solutions of (1.3) with  $u_\alpha(0) = \alpha$ ,  $u_\beta(0) = \beta$ , then there exists an integer  $n_0$  such that  $\lambda_1^{n_0-1}\alpha < \beta \leq \lambda_1^{n_0}\alpha$ , here  $\lambda_1 \in (1, \lambda)$  and  $\lambda$  is as in Theorem 3.1. By using Theorem 3.1  $n_0$  times, we get the results we want.

Q.E.D.

**Proof of Theorem 1(ii):** Let  $u_\alpha(r)$ ,  $u_\beta(r)$ ,  $w(t)$ ,  $Q(t)$ ,  $f(t)$  and  $g(t)$  be as in Theorem 3.1.

Suppose that Theorem 1(ii) does not hold. Then there exists  $T_1$ , such that  $Q(t)$  does not change sign when  $t > T_1$ . With no loss of generality, we assume  $Q(t) > 0$  for  $t > T_1$ . Since  $Q(t) \rightarrow 0$  as  $t \rightarrow \infty$ , we may assume  $Q'(T_1) < 0$ .

Since  $g(t) > 0$ , from (2.12) we have that

$$Q'' + f(t)Q' < 0 \quad t > T_1. \quad (3.13)$$

Multiplying (3.13) by  $\exp\left(\int_0^t f(s)ds\right)$  and integrating over  $(\tau, t)$ ,  $T_1 < \tau < t$ , we have that

$$\exp\left(\int_0^t f(s)ds\right)Q'(t) < \exp\left(\int_0^\tau f(s)ds\right)Q'(\tau) \quad T_1 \leq \tau < t. \quad (3.14)$$

Letting  $\tau = T_1$  in (3.14), we conclude that  $Q'(t) < 0$  for all  $t > T_1$ .

By Theorem B (and Remark 1.2),  $(w^p - w)/(w - 1) \rightarrow (p - 1)$ ,  $k(t)v_\alpha^{p-1} \rightarrow L^{p-1}$  as  $t \rightarrow \infty$ , and  $b_0^2 - 4c_0 < 0$  when  $(n + 2 + 2l)/(n - 2) < p < p_c$ , there exist constants  $\tilde{b}_0$  and  $\tilde{c}_0$ , and  $T_2 > T_1$ , such that  $\tilde{b}_0 > f(t)$ ,  $\tilde{c}_0 < g(t)$  and  $\tilde{b}_0^2 - 4\tilde{c}_0 < 0$ , when  $t > T_2$ .

Consider equation

$$q'' + \tilde{b}_0q' + \tilde{c}_0q = 0. \quad (3.15)$$

By O.D.E. theory, any solution  $q(t)$  of (3.15) is oscillatory; so there exist  $s_2 > s_1 > T_2$ , such that  $q(s_1) = q(s_2) = 0$ ,  $q(t) > 0$  on  $(s_1, s_2)$ . Multiplying (3.4) by  $q$  and (3.15) by  $Q$ , we have

$$qQ'' + f(t)qQ' + g(t)qQ = 0 \quad (3.16)$$

and

$$Qq'' + \tilde{b}_0Qq' + \tilde{c}_0Qq = 0. \quad (3.17)$$

Subtracting (3.17) from (3.16) yields

$$(Q'q - Qq')' + f(t)qQ' - \tilde{b}_0Qq' + (g(t) - \tilde{c}_0)qQ = 0 \quad s_1 < t < s_2. \quad (3.18)$$

Since  $f(t) < \tilde{b}_0$  and  $g(t) > \tilde{c}_0$ , we have

$$(Q'q - Qq')' + \tilde{b}_0(qQ' - Qq') < 0 \quad s_1 < t < s_2, \quad (3.19)$$

multiplying (3.19) by  $e^{\tilde{b}_0 t}$  and integrating over  $(s_1, s_2)$ , we have

$$e^{\tilde{b}_0 s_2}(Q'q - Qq')(s_2) < e^{\tilde{b}_0 s_1}(Q'q - Qq')(s_1). \quad (3.20)$$

This implies  $q'(s_2) > q'(s_1)$ , that is impossible since  $q'(s_2) < 0 < q'(s_1)$ . The contradiction completes our proof.

Q.E.D.

## 4 Singular Solutions

In this section, we will give the proof of Theorem 2. In the case of that  $K(r) = |x|^l$ ,  $l > -2$ , problem (1.1) reduces to

$$\Delta u + |x|^l u^p = 0. \quad (4.1)$$

The singular solutions of (4.1) are known quite well. For the sake of completeness, here we state the following results (see [GS1] and [W].)

(i) when  $m < n - 2$ , (4.1) has a singular solution

$$U_s(r) = Lr^{-m}, \quad (4.2)$$

(ii) when  $(n - 2)/2 < m < n - 2$ , all singular solutions of (4.1) consists of  $U_s(r)$  and another family of singular solutions  $\{u_\lambda\}$ ; and

$$\lim_{r \rightarrow 0^+} \frac{U_s(r)}{u_\lambda(r)} = 1, \quad u_\lambda(r) = O(r^{n-2}) \text{ at } r = \infty, \quad (4.3)$$

(iii) when  $m = (n - 2)/2$ , all singular solutions consists of  $U_s(r)$  and a family of singular solutions  $\{u_\lambda(r)\}$  oscillating around  $U_s(r)$  both at  $r = 0$  and  $r = \infty$ .

(iv) when  $m < (n - 2)/2$ ,  $U_s(r)$  is the only singular solution.

The essential property of singular solutions of (1.3) is the “blow-up” rate at  $r = 0$ . Our first result is concerning to this property. Roughly speaking, we will show that the only possible ”blow-up” rate is  $r^{-m}$ .

**Theorem 4.1** . Suppose  $l > -2$ ,  $m < n - 2$ ,  $u(r)$  is a positive solution of (1.3). If  $K(r)$  satisfies

(i)  $(K.1')$  and  $(K.2)$ , if  $0 < m < (n - 2)/2$ , or

(ii)  $(K.1')$  and  $(K.3)$ , if  $(n - 2)/2 < m < n - 2$ ,

then

$$\lim_{r \rightarrow 0^+} r^m u(r) \equiv u_0 = \begin{cases} \frac{L}{k_0^{\frac{1}{p-1}}}, & \text{or} \\ 0. \end{cases}$$

Furthermore, if  $u_0 = 0$ , then  $u(r)$  is bounded at  $r = 0$ , hence is a regular solution.

**Proof:** Consider Kelvin's transformation, i.e.

$$w(s) = s^{-(n-2)}u(s^{-1}) \quad s > 0,$$

then  $w(s)$  satisfies

$$w'' + \frac{n-1}{s}w' + K(s^{-1})s^{p(n-2)-(n+2)}w^p = 0. \quad (4.4)$$

Let  $\tilde{K}(s) = K(s^{-1})s^{p(n-2)-(n+2)}$ ,  $l' = p(n-2) - (n+2) - l$ , then  $m' = (n-2) - m$  and  $\tilde{K}(s)s^{-l'} = k(s^{-1})s^l$ . By Theorem B, we have

$$\lim_{r \rightarrow 0^+} r^m u(r) = \lim_{s \rightarrow \infty} s^m w(s) \equiv \begin{cases} \frac{L}{k_0^{\frac{1}{p-1}}}, & \text{or} \\ 0. \end{cases}$$

If the limit is 0, again, by Theorem B we know  $\lim_{r \rightarrow 0^+} u(r) = \lim_{s \rightarrow \infty} s^{n-2}w(s)$  exists and is positive. Thus we finish the proof.

Q.E.D.

**Corollary 4.1** . Let  $u(r)$  be a positive solution of equation (1.3),  $K(r)$  as in Theorem 4.1. If  $r = 0$  is a nonremovable singular point, then

$$\lim_{r \rightarrow 0^+} r^m u(r) = \frac{L}{k_0^{\frac{1}{p-1}}}.$$

**Lemma 4.1** . Suppose  $K$  as in Theorem 4.1. Let  $u(r)$  be a positive solution of (1.3), then

$$\lim_{r \rightarrow 0^+} r \frac{d}{dr} (r^m u(r)) = 0.$$

**Proof:** As in Lemma 2.2, let  $v(t) = r^m u(r)$ ,  $r = e^t$ , then

$$v'' + b_0 v' - L^{p-1} v + k(t) v^p = 0,$$

or

$$(e^{b_0 t} v'(t))' + e^{b_0 t} (k(t) v^p - L^{p-1} v) = 0.$$

Integrating from  $\tau$  to  $t$  for  $\tau < t$ , we have

$$e^{b_0 t} v'(t) - e^{b_0 \tau} v'(\tau) + \int_{\tau}^t e^{b_0 s} (k(s)v^p - L^{p-1}v) ds = 0.$$

By Theorem 4.1, we know that  $\lim_{t \rightarrow -\infty} (k(t)v^p - L^{p-1}v) = 0$ . Given  $\varepsilon > 0$ , there exists  $t_\varepsilon$ ,  $k(t)v^p - L^{p-1}v < \varepsilon$  when  $t < t_\varepsilon$ , and

$$|v'(t)| \leq e^{b_0(\tau-t)} v'(\tau) + \frac{\varepsilon}{b_0} (-e^{-b_0(t-\tau)} + 1).$$

Letting  $\tau$  (if necessary, take a subsequence) go to  $-\infty$ , we have

$$|v'(t)| \leq \frac{\varepsilon}{b_0}$$

if  $t < t_\varepsilon$ . Since  $\varepsilon$  can be arbitrary small, we conclude  $\lim_{t \rightarrow -\infty} v'(t) = 0$ , or equivalently  $\lim_{r \rightarrow 0^+} r \frac{d}{dr} (r^m u(r)) = 0$ .

Q.E.D.

**Corollary 4.2** . Suppose  $K$  is as in Theorem 4.1,  $u(r)$  is radial positive singular solution of (1.3), then

$$\lim_{r \rightarrow 0^+} r^{m+1} u'(r) = -m \frac{L}{k_0^{\frac{1}{p-1}}}.$$

**Proof:** Recall that

$$\begin{aligned} v'(t) &= r \frac{d}{dt} (r^m u(r)) \\ &= mr^m u(r) + r^{m+1} u'(r) \end{aligned}$$

The corollary holds since the results of Lemma 4.1 and Theorem 4.1.

Q.E.D.

Next we will show the uniqueness of singular solution. The special case when  $l = 0$  is also treated by Janson, Pan and Yi in [JPY].

**Lemma 4.2** . Suppose  $f(t)$  and  $g(t)$  are continuous functions,  $\lim_{t \rightarrow \infty} f(t) = b > 0$ ,  $\lim_{t \rightarrow \infty} g(t) = c > 0$ . Let  $y(t)$  be a solution of

$$y'' - f(t)y' + g(t)y = 0.$$

Then  $y(t)$  is unbounded as  $t \rightarrow \infty$ .

**Proof:** Let  $z = y'$ ,  $\mathbf{X} = \begin{pmatrix} z \\ y \end{pmatrix}$ . Then  $\mathbf{X}$  satisfies

$$\begin{aligned} \mathbf{X}' &= \begin{pmatrix} f(t) & -g(t) \\ 1 & 0 \end{pmatrix} \mathbf{X} \\ &= \begin{pmatrix} b & -c \\ 1 & 0 \end{pmatrix} \mathbf{X} + \begin{pmatrix} f(t) - b & c - g(t) \\ 0 & 0 \end{pmatrix} \mathbf{X} \\ &\equiv \mathbf{M}\mathbf{X} + \mathbf{N}(t)\mathbf{X}. \end{aligned} \tag{4.5}$$

Let  $\Psi(t)$  be a characteristic solution of system

$$\mathbf{X}' = \mathbf{M}\mathbf{X}, \tag{4.6}$$

then  $\Psi(t)$  is in one of the following three forms:

$$\begin{pmatrix} a_{11}e^{\lambda t} & a_{12}te^{\lambda t} \\ a_{21}e^{\lambda t} & a_{22}te^{\lambda t} \end{pmatrix}, \begin{pmatrix} a_{11}e^{\lambda_1 t} & a_{12}e^{\lambda_2 t} \\ a_{21}e^{\lambda_1 t} & a_{22}e^{\lambda_2 t} \end{pmatrix}, \begin{pmatrix} a_{11}e^{\lambda t} \cos \beta t & a_{12}e^{\lambda t} \sin \beta t \\ a_{21}e^{\lambda t} \sin \beta t & a_{22}e^{\lambda t} \cos \beta t \end{pmatrix}.$$

Here  $\lambda, \lambda_i > 0$ , and  $\beta$  is a constant. Without loss of generality, we may assume that  $a_{ii} \neq 0$ ,  $i = 1, 2$ . The differential equation (4.5) is equivalent to the integral equation

$$\mathbf{X}(t) = \Psi(t)C + \Psi(t) \int_{t_0}^t \Psi^{-1}(s)\mathbf{N}(s)\mathbf{X}(s)ds \tag{4.7}$$

where constant vector  $C$  is the initial value  $\Psi^{-1}\mathbf{X}(t_0)$ .

Denote  $\Psi^{-1}\mathbf{X}$  by  $\Phi(t)$ ,  $\Psi^{-1}\mathbf{N}\Psi$  by  $\mathbf{G}$ , then

$$\Phi(t) = C + \int_{t_0}^t \mathbf{G}(s)\Phi(s)ds.$$

Differentiating the above equation, we have that

$$\Phi'(t) = \mathbf{G}(t)\Phi(t),$$

thus

$$\Phi(t) = C \exp\left(\int_{t_0}^t \mathbf{G}(s)ds\right)$$

or

$$\mathbf{X}(t) = C\Psi(t) \exp\left(\int_{t_0}^t \mathbf{G}(s)ds\right).$$

Since  $\mathbf{N}(t) \rightarrow \begin{pmatrix} \circ(1) & \circ(1) \\ 0 & 0 \end{pmatrix}$  as  $t \rightarrow \infty$ , for any given  $\varepsilon > 0$ , there exists  $t_\varepsilon > 0$ , such that  $\mathbf{G}(t) = (g_{ij}(t))$  with  $|g_{ii}(t)| < \varepsilon$ ,  $i = 1, 2$ , when  $t > t_0 > t_\varepsilon$ . Hence  $\exp\left(\int_{t_0}^t \mathbf{G}(s)ds\right) = \mathbf{I} + (\tilde{g}_{ii}(t)(t - t_0))_{2 \times 2}$ , for some  $\tilde{g}_{ii}(t)$ , such that  $|\tilde{g}_{ii}(t)| < \varepsilon$ , and  $\mathbf{I}$  is the unit matrix. By choosing  $\varepsilon < \operatorname{Re}\left\{\frac{b_0 - \sqrt{b_0^2 - 4c}}{2}\right\}$  (this is always possible since  $c > 0$ ), we have that  $\|\mathbf{X}(t)\| \rightarrow \infty$  for  $t \rightarrow \infty$ . This ends our proof.

Q.E.D.

**Corollary 4.3** . Suppose  $K$  satisfies  $(K.1')$  and  $(K.2)$ ,  $p > \frac{n+2+2l}{n-2}$  (i.e.  $m < \frac{n-2}{2}$ ), then the radial positive singular solution of equation (1.3) is unique.

**Proof:** Suppose  $u_1(r)$  and  $u_2(r)$  are two different singular solutions. As we did in Section 3, let  $v_i(t) = r^m u_i(r)$ ,  $i = 1, 2$ ,  $r = e^t$ . Let  $w(t) = v_2(t)/v_1(t)$ , then we have

$$w'' + \left(b_0 + \frac{2v_1'}{v_1}\right)w' + k(t)v_1^{p-1}(w^p - w) = 0$$

let  $f(t) = b_0 + \frac{2v_1'}{v_1}$ ,  $g(t) = k(t)v_1^{p-1}\frac{w^p - w}{w-1}$  when  $w \neq 1$ , and  $g(t) = (p-1)k(t)v_1^p$  when  $w = 1$ ,  $Q(t) = w(-t) - 1$ . Then  $Q(t)$  satisfies

$$Q'' - f(-t)Q' + g(-t)Q = 0.$$

Suppose  $Q \not\equiv 0$ . By Corollary 4.1 and Lemma 4.1,  $\lim_{t \rightarrow \infty} Q(t) = 0$ ,  $\lim_{t \rightarrow \infty} Q'(t) = 0$ ,  $\lim_{t \rightarrow \infty} f(-t) = b_0$  and  $\lim_{t \rightarrow \infty} g(-t) = c_0$ . But by Lemma 4.2,  $(Q'(t), Q(t))$  is unbounded as  $t \rightarrow \infty$ . The contradiction shows the uniqueness of singular solution.

Q.E.D.

In the following we show the existence of singular solution of (1.3) by employing of Theorem 1 for  $p > p_c$ . By Theorem A, for any  $\alpha > 0$ , there exists a positive solution  $u_\alpha(r)$  of (1.3). By Theorem 1,  $u_\alpha(r)$  is strictly increasing in  $\alpha$ , and by Lemma 2.2  $r^m u_\alpha(r) < \frac{L}{[r^{-l}K(r)]^{\frac{1}{p-1}}}$ . From (1.4) we have that

$$\left( r^{n-1} u_\alpha'(r) \right)' + r^{n-1} K(r) u_\alpha^p(r) = 0.$$

Integrating from 0 to  $t$ , we have that

$$u_\alpha'(r) = -\frac{1}{r^{n-1}} \int_0^r s^{n-1} K(s) u_\alpha^p(s) ds.$$

Hence  $u_\alpha'$  is uniformly bounded on any compact subset of  $0 < r < \infty$  in  $\alpha$ , consequently,  $u_\alpha$  are uniformly continuous. Let  $U(r) = \lim_{\alpha \rightarrow \infty} u_\alpha(r)$ , so  $U(r)$  is well defined and is continuous on  $0 < r < \infty$ . By Lemma 2.2,

$$r^m u_\alpha(r) < r^m U(r) \leq \frac{L}{[r^{-l}K(r)]^{\frac{1}{p-1}}}.$$

Let  $B_{R,\rho}$  denote the region  $\rho < r < R$ . Consider the following boundary problem

$$\begin{cases} \Delta u + K(r)U^p = 0 \\ u|_{\partial B_{R,\rho}} = U. \end{cases}$$

We want to show  $u = U$ . For each  $\alpha > 0$ , let  $u_\alpha$  be the solution of (1.3), so we have

$$\Delta(u - u_\alpha) = -K(r)(U^p - u_\alpha^p) < 0.$$

By maximum principle we have  $u - u_\alpha > 0$ , hence  $u - U \geq 0$  in  $B_{R,\rho}$  since  $u - u_\alpha > 0$  on  $\partial B_{R,\rho}$ . Let  $\phi_\varepsilon = \varepsilon e^r$ . Then  $\phi_\varepsilon$  satisfies

$$\Delta\phi_\varepsilon = \varepsilon e^r \left(1 + \frac{n-1}{r}\right) > 0.$$

Thus for any fixed  $R$ ,  $\rho$  and  $\varepsilon$ , when  $\alpha$  large enough,  $\Delta(u - u_\alpha + \phi_\varepsilon) > 0$  in  $B_{R,\rho}$ . Again, by maximum principle, we have

$$u - u_\alpha + \phi_\varepsilon \leq (u - u_\alpha + \varepsilon e^r)|_{\partial B_{R,\rho}}, \quad \rho < r < R.$$

Letting  $\alpha$  go to  $\infty$  first, then  $\varepsilon$  go to 0, we get  $u - U \leq 0$ , therefore  $u = U$  in  $B_{R,\rho}$ . Since  $R$  and  $\rho$  can be arbitrary, we get  $u = U$  on  $0 < r < \infty$ .

Combining this results and Corollaries 4.1, 4.2 and 4.3, we complete the proof of Theorem 2.

Q.E.D.

## 5 Asymptotic Expansion At Infinity

In this section, we will extend the expansion results for  $K \equiv 1$  obtained by Gui, Ni and Wang in [GNW] to our more general  $K$  assumed in Theorem 1. The techniques are first developed by Li in [L1].

Let  $u$  be a solution of (1.3) such that  $\lim_{r \rightarrow \infty} r^m u(r) = u_\infty \equiv \frac{L}{k_\infty^{\frac{1}{p-1}}}$ . Let  $w(t) = r^m u(r) - u_\infty$ ,  $r = e^t$  then  $w$  satisfies

$$w'' + b_0 w' - L^{p-1}(u_\infty + w) + k(t)(u_\infty + w)^p = 0, \quad (5.1)$$

where  $k(t)$  is given in Lemma 2.2 and  $b_0$ ,  $L$  is given in (1.4). Let  $g(\tau) = (u_\infty + \tau)^p - u_\infty^p - pu_\infty^{p-1}\tau$ , then  $g$  has expansion

$$g(w) = d_2 w^2 + \cdots + d_M w^M + O(w^{M+1}) \quad (5.2)$$

at  $w = 0$  for any positive integer  $M \geq 2$ . Where  $d_i$ ,  $i = 2, \dots, M$ , depend only on  $p, n$ , and  $l$ , and  $d_2 = \frac{p(p-1)}{2} u_\infty^{p-2} > 0$ . Let  $\varphi(t) = u_\infty^p (k(t) - k_\infty)$ ,  $d_1 = pu_\infty^{p-1}$ ,  $\tilde{g}(w) = d_1 w + g(w)$ .

Denote

$$G(t, w) = \varphi(t) + (k(t) - k_\infty)\tilde{g}(w) + k_\infty g(w),$$

then (5.1) becomes

$$w'' + b_0 w' + c_0 w + G(t, w) = 0 \quad (5.3)$$

Since  $p > p_c$ , the characteristic equation of (5.3) has two negative roots

$$-\lambda_2 < -\lambda_1 < 0.$$

Furthermore, we assume that there exists some positive constant  $\gamma > 0$  such that

$$\varphi(t) = u_\infty^p (k(t) - k_\infty) = O(e^{-\gamma t}) \quad \text{at } t = \infty. \quad (5.4)$$

Recall that  $w'(t) = mr^m u(r) + r^{m+1} u'(r)$ ,  $r = e^t$ , and that

$$r^{m+1} u'(r) = -\frac{1}{r^{n-m-2}} \int_0^r s^{n-1} K(s) u^p(s) ds.$$

By (K.1) and Theorem B(i) we have that, as  $r \rightarrow \infty$

$$r^{m+1} u'(r) = O(1),$$

Thus  $w'(t)$  is bounded at  $t = \infty$ .

Let  $G_1(t) = \int_0^t g(s) ds$ . Multiplying (5.3) by  $w'$  and integrating from  $t$  to  $T > t$ ,

$$\begin{aligned} w'^2(T) + 2b_0 \int_t^T w'^2 ds + c_0 w^2(T) + 2 \int_t^T w' \varphi(t) \left[1 + \frac{1}{u_\infty^p} \tilde{g}(w)\right] ds \\ + 2k_\infty G_1(w(T)) = w'^2 + c_0 w^2 + 2k_\infty G_1(w). \end{aligned}$$

Let  $T$  (a sequence if necessary) go to  $\infty$ , we have

$$\int_t^\infty w'^2 ds \leq C(w'^2 + w^2 + G_1(w) + \int_t^\infty |\varphi| ds) \quad (5.5)$$

for some constant  $C$ . Hence  $w'^2 \in L^1(T, \infty)$ .

Multiplying (5.3) by  $w$  and integrating from  $t$  to  $T > t$ ,

$$\begin{aligned} ww'(T) + \frac{b_0}{2} w^2 + c_0 \int_t^T w^2 ds + \int_t^T wG(s, w) ds \\ = ww'(t) + \int_t^T w'^2 ds + \frac{b_0}{2} w^2. \end{aligned}$$

Since  $w \rightarrow 0$  as  $t \rightarrow \infty$ , there exists  $T_0 > 0$  that  $k_\infty w g(w) < \frac{c_0}{2} w^2$  when  $t > T_0$ . Letting  $T$  (again, taking a sequence if necessary) go to  $\infty$ , we have that

$$\int_t^\infty w^2 ds \leq C(w^2 + \int_t^\infty w'^2 ds + \int_t^\infty |\varphi| ds) \quad (5.6)$$

for some constant  $C$ . Combining (5.5) and (5.6) we conclude that for any positive integer  $i$ ,

$$\int_{t_i}^\infty \dots \int_{t_1}^\infty w'(s)^2 ds \dots dt_{i-1} < \infty.$$

This is equivalent to

$$t^i w'^2 \in L^1(T, \infty). \quad (5.7)$$

A direct consequence of (5.7) is that  $w' \in L^1(T, \infty)$  by letting  $i = 2$  and using Hölder inequality.

We will deal with the cases of  $\gamma \leq \lambda_1$  and  $\gamma > \lambda_2$  separately.

**Case 1°**  $0 < \gamma \leq \lambda_1$ .

Let  $R(t) = e^{(\gamma-\varepsilon)t} w(t)$ ,  $\varepsilon \in (0, \gamma)$ . Then  $R(t)$  is a solution of the following equation

$$R'' + (b_0 - 2(\gamma - \varepsilon))R' + b(\gamma, \varepsilon)R + e^{(\gamma-\varepsilon)t} G(t, w) = 0.$$

Where  $b(\gamma, \varepsilon) \equiv \gamma^2 - b_0\gamma + c_0 + \varepsilon(b_0 + \varepsilon - 2\gamma) > 0$ . Multiplying above equation by  $2R'$  and integrating over  $(T, t)$

$$\begin{aligned} R'^2(t) + 2(b_0 - 2(\gamma - \varepsilon)) \int_T^t R'^2 ds + b(\gamma, \varepsilon)R^2(t) + \\ 2 \int_T^t e^{(\gamma-\varepsilon)s} R'(s)G(s, w(s)) ds = R'^2(T) + b(\gamma, \varepsilon)R^2(T). \end{aligned} \quad (5.8)$$

Since  $w'(t)$  is bounded at  $t = \infty$ , if choose  $\varepsilon > \frac{\gamma}{2}$ , then  $e^{(\gamma-\varepsilon)t} R'(t)[\varphi(t) + (k(t) - k_\infty)\tilde{g}(w)] \in L^1(T, \infty)$ . Hence, integrating by parts, from (5.8) we conclude

$$b(\gamma, \varepsilon)R^2(t) + \frac{g(w)}{w}R^2(t) - \int_T^t \frac{d}{ds} \left( \frac{g(w(s))}{w(s)} \right) R^2(s) ds \leq C(T)$$

for some constant  $C(T)$ .

We claim for large  $T$ , we have

$$R^2(t) \leq \frac{2}{b(\gamma, \varepsilon)} C(T)$$

holding uniformly for all  $t > T$ .

In fact, since  $w' \in L^1(0, \infty)$ , there exists  $T_0$  such that if  $t > T_0$ , then  $\frac{g(w)}{w} > 0$  (since  $d_2 > 0$ ) and

$$\int_T^\infty \left| \frac{d}{ds} \left( \frac{g(w(s))}{w(s)} \right) \right| ds < \frac{b(\gamma, \varepsilon)}{2}$$

for  $T > T_0$ .

On the contrary, if there exists  $t_0 > T > T_0$  such that  $R^2(t) < R^2(t_0) = \frac{2}{b(\gamma, \varepsilon)}C(T)$  for  $T_0 < t < t_0$ , then we get

$$\left( b(\gamma, \varepsilon) + \frac{g(w)}{w} - \int_T^\infty \left| \frac{d}{ds} \left( \frac{g(w(s))}{w(s)} \right) \right| ds \right) R^2(t_0) \leq C(T).$$

From this we derive

$$R^2(t_0) < \frac{2}{b(\gamma, \varepsilon)}C(T),$$

which is a contradiction. It follows that  $R$  is bounded at  $t = \infty$ , and from (5.8) we get

$$|w| + |w'| = O(e^{-(\gamma-\varepsilon)t}). \quad (5.9)$$

Since  $w$  is a solution of (5.3), we can write  $w$  as follows ( see [H])

$$w(t) = ae^{-\lambda_1 t} + be^{-\lambda_2 t} + \frac{1}{\lambda_2 - \lambda_1} \int_T^t (e^{\lambda_2(s-t)} - e^{\lambda_1(s-t)}) G(s, w(s)) ds \quad (5.10)$$

for some constants  $a$  and  $b$ . By the definition of  $G$ , we get

$$\begin{aligned} w(t) &= ae^{-\lambda_1 t} + be^{-\lambda_2 t} + \psi_1(t, T) + \\ &\quad \frac{1}{\lambda_2 - \lambda_1} \int_T^t (e^{\lambda_2(s-t)} - e^{\lambda_1(s-t)}) [(k(s) - k_\infty)\tilde{g}(w) + k_\infty g(w)] ds, \end{aligned} \quad (5.11)$$

where

$$\begin{aligned} \psi_1(t, T) &= \frac{1}{\lambda_2 - \lambda_1} \int_T^t (e^{\lambda_2(s-t)} - e^{\lambda_1(s-t)}) \varphi(s) ds \\ &= \begin{cases} O(e^{-\gamma t}) & \text{if } \gamma < \lambda_1, \\ O(te^{-\gamma t}) & \text{if } \gamma = \lambda_1. \end{cases} \end{aligned}$$

Bringing (5.9) into (5.11) we get

$$w(t) = O(e^{-2(\gamma-\varepsilon)t}).$$

Now, we take  $\varepsilon \in (\frac{\gamma}{2}, \frac{3\gamma}{4})$ , and let  $\theta = 3\gamma - 4\varepsilon$ , then  $E_1(t, w) \equiv G(t, w) - \varphi(t) = (k(t) - k_\infty)\tilde{g}(w) + k_\infty g(w) = O(e^{-(\gamma+\theta)t})$ . Without loss of generality, we assume that  $\theta \notin \text{span}\{\gamma, \lambda_1, \lambda_2\}$  over  $\mathbb{Z}$ . Thus, from (5.11) we get

$$w(t) = \begin{cases} \psi_1(t, T) + O(e^{-(\gamma+\theta)t}) & \text{if } \lambda_1 > \gamma + \theta, \\ \psi_1(t, T) + a_1 e^{-\lambda_1 t} + O(e^{-(\gamma+\theta)t}) & \text{if } \lambda_1 < \gamma + \theta < \lambda_2, \\ \psi_1(t, T) + a_1 e^{-\lambda_1 t} + b_1 e^{-\lambda_2 t} + O(e^{-(\gamma+\theta)t}) & \text{if } \lambda_2 < \gamma + \theta. \end{cases} \quad (5.12)$$

It is worthy of noting that while dealing with the calculations above, we break up the integrals  $\int_T^t e^{\lambda_i s} E_1(s, w) ds$  into two parts  $\int_T^\infty - \int_t^\infty$ , once  $\lambda_i < \theta + \gamma$ , and

$$a_1 = a - \frac{1}{\lambda_2 - \lambda_1} \int_T^\infty e^{\lambda_1 s} E_1(s, w) ds$$

and

$$b_1 = b + \frac{1}{\lambda_2 - \lambda_1} \int_T^\infty e^{\lambda_2 s} E_1(s, w) ds.$$

Before giving the general expansion form of  $w$  at  $t = \infty$ , we carry our calculations one more to make the process more clear. For example, we deal with the case  $\lambda_1 > \gamma + \theta$ . Define  $E_2(t, w) \equiv E_1(t, w) - (d_1(k(t) - k_\infty)\psi_1(t, T) + d_2 k_\infty \psi_1^2) = O(e^{-(2\gamma+\theta)t})$ . Bring (5.12) into (5.11) we have

$$\begin{aligned} w(t) &= a e^{-\lambda_1 t} + b e^{-\lambda_2 t} + \psi_1(t, T) + \psi_2(t, T) \\ &\quad + \frac{1}{\lambda_2 - \lambda_1} \int_T^t (e^{\lambda_2(s-t)} - e^{\lambda_1(s-t)}) E_2(s, w) ds \\ &= \begin{cases} \psi_1(t, T) + \psi_2(t, T) + O(e^{-(2\gamma+\theta)t}), & \text{if } \lambda_1 > 2\gamma + \theta, \\ \psi_1(t, T) + \psi_2(t, T) + a_1 e^{-\lambda_1 t} + O(e^{-(2\gamma+\theta)t}), & \text{if } \lambda_1 < 2\gamma + \theta < \lambda_2, \\ \psi_1(t, T) + \psi_2(t, T) + a_1 e^{-\lambda_1 t} + b_1 e^{-\lambda_2 t} + O(e^{-(2\gamma+\theta)t}), & \text{if } \lambda_2 < 2\gamma + \theta, \end{cases} \end{aligned}$$

where

$$\psi_2(t, T) = \frac{1}{\lambda_2 - \lambda_1} \int_T^t (e^{\lambda_2(s-t)} - e^{\lambda_1(s-t)}) (d_1(k(s) - k_\infty)\psi_1(s, T) + d_2k_\infty\psi_1^2) ds,$$

$$a_1 = a - \frac{1}{\lambda_2 - \lambda_1} \int_T^\infty e^{\lambda_1 s} E_2(s, w) ds,$$

and

$$b_1 = b + \frac{1}{\lambda_2 - \lambda_1} \int_T^\infty e^{\lambda_2 s} E_2(s, w) ds.$$

Suppose that,  $k_i, i = 1, 2$ , are the positive integers, such that,  $(k_i - 1)\gamma < \lambda_i \leq k_i\gamma$ . For such  $k_i$ , we can choose  $\theta$  by adjusting  $\varepsilon$  in such way that  $(k_i - 1)\gamma < \lambda_i < k_i\gamma + \theta$ . Generally, by calculations similar to the previous, we have the following expansion after the  $k_2$ th iteration

$$w(t) = \psi_1(t, T) + \cdots + \psi_{k_2}(t, T) + \sum_{\substack{j \in I_i \\ i < (k_2\gamma + \theta)/\lambda_1}} a_{ij}(t) e^{-i\lambda_1 t} + b_1 e^{-\lambda_2 t} + O(e^{-(k_2\gamma + \theta)}), \quad (5.13)$$

where

$$I_i = \{j \in N \mid j\gamma + i\lambda_1 < k_2\gamma + \theta\},$$

and,  $a_{10}(t) = a_1$  and  $b_1$  are constants. If  $\lambda_1 \neq k_1\gamma$ ,  $\lambda_2 \neq k_2\gamma$ ,  $\psi_i(t, T) = O(e^{-i\gamma t})$ , and  $a_{ij}(t) = O(e^{-j\gamma t})$  depending only on  $a_1$  and  $\psi_1$ ; if  $\lambda_1 = k_1\gamma, \lambda_2 < k_2\gamma$ , then  $\psi_i(t, T) = O(e^{-i\gamma t})$ , for  $i < k_1$ , and  $\psi_i(t, T) = O(te^{-i\gamma t})$  for  $k_1 \leq i < k_2$ , and  $a_{ij}(t) = O(e^{-j\gamma t})$  for  $j < k_1$ ,  $a_{ij}(t) = O(te^{-j\gamma t})$  for  $k_1 \leq j \leq k_2$ ; if  $\lambda_1 = k_1\gamma, \lambda_2 = k_2\gamma$ , then  $\psi_i(t, T) = O(e^{-i\gamma t})$  when  $i < k_1$ ,  $\psi_i(t, T) = O(te^{-i\gamma t})$  when  $k_1 \leq i < k_2$ ,  $\psi_{k_2}(t, T) = O(t^2 e^{-i\gamma t})$ ,  $a_{ij}$  are the same as in the case  $\lambda_1 = k_1\gamma, \lambda_2 < k_2\gamma$ .

It is easy to see that all the coefficients of the terms before  $b_1 e^{-\lambda_2 t}$  are determined once  $a_1$  is fixed. Keeping this procedure and back to our original variable  $r$ , without discriminations we use the same notations as in (5.13),  $u$  has expansion of the following form

$$u(r) = \frac{1}{r^m} \sum_{i=1}^{k_2} \psi_i(r) + \sum_{\substack{j \in I_i \\ i < (k_2\gamma + \theta)/\lambda_1}} \frac{a_{ij}(r)}{r^{m+i\lambda_1}} + \frac{b_1}{r^{m+\lambda_2}} + \cdots + O(r^{-(n-2+\varepsilon)}) \quad (5.14)$$

at  $r = \infty$  for some  $\varepsilon > 0$ , where  $a_{10}(r) \equiv a_1$  and  $b_1$  are constants.

**Case 2°**  $\gamma > \lambda_1$ .

For simplicity, we assume that  $\gamma > \lambda_2$ . Let  $R(t) = e^{(\lambda_1 - \varepsilon)t}w(t)$ ,  $\varepsilon \in (0, \lambda)$ . Then  $R(t)$  is a solution of the following equation

$$R'' + (b_0 - 2(\lambda_1 - \varepsilon))R' + \varepsilon(b_0 + \varepsilon - 2\lambda_1)R + e^{(\lambda_1 - \varepsilon)t}G(t, w) = 0.$$

Where  $G(t, w)$  is defined in (5.3). Similar to *Case 1°*, we have that

$$|w| + |w'| = O(e^{-(\lambda_1 - \varepsilon)t}).$$

Again, using formula (5.10) we have that

$$\begin{aligned} w(t) &= ae^{-\lambda_1 t} + be^{-\lambda_2 t} + \psi_0(t, T) \\ &\quad \frac{1}{\lambda_2 - \lambda_1} \int_T^t (e^{\lambda_2(s-t)} - e^{\lambda_1(s-t)}) [(k(s) - k_\infty)\tilde{g}(w) + k_\infty g(w)] ds, \end{aligned} \tag{5.15}$$

where

$$\begin{aligned} \psi_0(t, T) &= \frac{1}{\lambda_2 - \lambda_1} \int_T^t (e^{\lambda_2(s-t)} - e^{\lambda_1(s-t)}) \varphi(s) ds \\ &= \frac{1}{\lambda_2 - \lambda_1} \left( \int_T^\infty - \int_t^\infty \right) \\ &= \tilde{a}_1 e^{-\lambda_1 t} + \tilde{b}_1 e^{-\lambda_2 t} - \frac{1}{\lambda_2 - \lambda_1} \int_t^\infty (e^{\lambda_2(s-t)} - e^{\lambda_1(s-t)}) \varphi(s) ds \\ &\equiv \tilde{a}_1 e^{-\lambda_1 t} + \tilde{b}_1 e^{-\lambda_2 t} + \psi_1(t), \end{aligned}$$

and  $\psi_1(t) = O(e^{-\gamma t})$  at  $t = \infty$ . Let  $\theta = 3\lambda_1 - 4\varepsilon$ . Again, we assume  $\theta$  can not expressed as a linear combination of  $\gamma$ ,  $\lambda_1$  and  $\lambda_2$  over  $\mathbb{Z}$ . Then

$$w(t) = \begin{cases} a_1 e^{-\lambda_1 t} + O(e^{-(\lambda_1 + \theta)t}), & \text{if } \lambda_2 > \lambda_1 + \theta \\ a_1 e^{-\lambda_1 t} + b_1 e^{-\lambda_2 t} + O(e^{-(\lambda_1 + \theta)t}), & \text{if } \lambda_2 < \lambda_1 + \theta < \gamma \\ a_1 e^{-\lambda_1 t} + b_1 e^{-\lambda_2 t} + \psi_1(t, T) + O(e^{-(\lambda_1 + \theta)t}), & \text{if } \gamma < \lambda_1 + \theta. \end{cases} \quad (5.16)$$

where  $a_1 = a + \tilde{a}_1$ ,  $b_1 = b + \tilde{b}_1$  and

$$\tilde{a}_1 = \frac{-1}{\lambda_2 - \lambda_1} \int_T^\infty e^{\lambda_1 s} \varphi(s) ds, \quad \tilde{b}_1 = \frac{1}{\lambda_2 - \lambda_1} \int_T^\infty e^{\lambda_2 s} \varphi(s) ds.$$

To make our expansion more clear, we repeat the iteration one more time. Consider the case  $\lambda_1 + \theta < \lambda_2 < 2\lambda_1 + \theta$ . Putting (5.16) into (5.15), we obtain

$$w(t) = \begin{cases} a_1 e^{-\lambda_1 t} + a_2 e^{-2\lambda_1 t} + b_1 e^{-\lambda_2 t} + O(e^{-(2\lambda_1 + \theta)t}) & \text{if } \lambda_2 \neq 2\lambda_1 \\ a_1 e^{-\lambda_1 t} + a_2 e^{-2\lambda_1 t} + c_1 t e^{-2\lambda_1 t} + b_1 e^{-\lambda_2 t} + O(e^{-(2\lambda_1 + \theta)t}) & \text{if } \lambda_2 = 2\lambda_1. \end{cases} \quad (5.17)$$

Keep doing in this way and back to the old variable  $r$ , like (5.14) we obtain

$$u(r) = \begin{cases} \frac{L}{k_\infty \frac{1}{p-1} r^m} + \frac{a_1}{r^{m+\lambda_1}} + \frac{a_2}{r^{m+2\lambda_1}} + \dots + \frac{b_1}{r^{m+\lambda_2}} + \dots + O\left(\frac{1}{r^{n-2+\varepsilon}}\right), & \text{if } \lambda_2 \neq \Lambda\lambda_1 \\ \frac{L}{k_\infty \frac{1}{p-1} r^m} + \frac{a_1}{r^{m+\lambda_1}} + \frac{a_2}{r^{m+2\lambda_1}} + \dots + \frac{c_1 \log r}{r^{m+\Lambda\lambda_1}} + \frac{b_1}{r^{m+\lambda_2}} & \\ + \dots + O\left(\frac{1}{r^{n-2+\varepsilon}}\right), & \text{if } \lambda_2 = \Lambda\lambda_1 \end{cases} \quad (5.18)$$

for some positive integer  $\Lambda > 1$ . From the above calculations and discussions it follows

**Theorem 5.1** . Suppose  $p > p_c$ , and there exists  $\gamma > 0$  such that

$$r^{-l} K(r) - k_\infty = O\left(\frac{1}{r^\gamma}\right) \text{ at } r = \infty.$$

Let  $u$  be a solution of (1.3) satisfying  $\lim_{r \rightarrow \infty} r^m u(r) = \frac{L}{k_\infty \frac{1}{p-1}}$ . Then  $u$  has an expansion at  $r = \infty$ , which, in particular, is (5.14) if  $\gamma \leq \lambda_1$ , or (5.18) if  $\gamma > \lambda_2$ .

**Remark 5.1** . For the case that  $\gamma \in (\lambda_1, \lambda_2)$ , by a similar argument the solution  $u$  has an expansion which consisting some mixed terms between  $a_1 r^{-(m+\lambda_1)}$  and  $b_1 r^{-(m+\lambda_2)}$ , which are generated by  $a_1 r^{-(m+\lambda_1)}$  and  $\varphi(\log(r))$ . For a given solution  $u$ ,  $a_1 r^{-(m+\lambda_1)}$  and  $b_1 r^{-(m+\lambda_2)}$  are the two independent terms in its expansion at infinity.

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