Ph.D. Comprehensive Exam in Algebra

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Complete exactly three (3) numbered parts in each of the following sections: **Groups, Rings and Modules** and **Fields.**

Groups:

1. Describe all groups that have only one proper non-trivial subgroup, and justify your answer.
2. Prove that if G is a group of order pq with p and q primes such that $p > q$ and p is not congruent to $1\,(mod\; q)$, then G is cyclic.
3. Prove that every group of order 200 is solvable.
4. Up to isomorphism, find all abelian groups G with $G/T \approx \mathbb{Z} \oplus \mathbb{Z}$ and $|T| = 2800$ where T is the torsion subgroup of G. Justify your answer.
5. Let G be a group with $|G| = p^n m$ where p is prime and $(p, m) = 1$. Prove that G has a subgroup of order p^n.

Rings and modules: R is a ring with identity $1 \in R$.

1. Prove that a module M is artinian if and only if for each set of submodules $\{K_i \mid i \in I\}$ there is a finite subset $F \subset I$ such that $\bigcap_{i \in I} K_i = \bigcap_{i \in F} K_i$.
2. A submodule K of a R-module M is *superfluous* if for all submodules L of M, $K + L = M$ implies $L = M$.
   (a) Let $f : M \to N$ be an R-epimorphism with $K = \operatorname{Ker} f$. Prove that K is superfluous in M if and only if for all $g : W \to M$, if fg is an epimorphism, then g is an epimorphism.
   (b) The intersection of all maximal submodules of a module M is denoted $J(M)$. Prove that every superfluous submodule of M is contained in $J(M)$.
3. Up to isomorphism, classify all noncommutative semisimple rings of order 2^8. (You may use the fact that all finite division rings are fields.)
4. Prove that a PID is a UFD.
5. (a) Define "projective R-module" and "free R-module".
   (b) Show that a direct sum of projective R-modules is projective.
   (c) Give an example of a projective R-module that is not free. Justify your answer.

Fields:

1. Find the Galois group of $x^3 - 2 \in \mathbb{Q}[x]$. Give the correspondence between subgroups of G and the subfields of F, the splitting field of $x^3 - 2$ over $\mathbb{Q}$.

2. Explain what it means for a polynomial in $\mathbb{Q}[x]$ to be solvable by radicals over $\mathbb{Q}$. Explain why every fourth degree polynomial in $\mathbb{Q}[x]$ is solvable by radicals over $\mathbb{Q}$. Give an example, or explain why, a polynomial of degree five from $\mathbb{Q}[x]$ need not be solvable by radicals over $\mathbb{Q}$.

3. Describe up to isomorphism all finite fields. Justify your answer.

4. Let $K \subseteq E$ and $E \subseteq F$ be finite dimensional field extensions. Show that $[F : K] = [F : E][E : K]$.

5. Let F be a field.

   (a) Define *separable* polynomial $f(x) \in F[x]$.

   (b) Prove that if the characteristic of F is 0, then every polynomial is separable over F.