Ph.D. Conprehensive Exam in Algebra
August 28, 2002
Kent Fuller and Norman Johnson

Complete exactly three (3) numbered parts in each of the following sections: **Groups, Rings and Modules** and **Fields.**

Groups:

1. Let $f : G \to H$ be a group homomorphism. Prove that if $a \in G$ has finite order, then the order of $f(a)$ divides the order of a.

2. Prove that every group of order 200 is solvable.

3. Prove that if G is a finite group with p-sylow subgroup P, then any subgroup $H \subseteq G$ such that $N_G(P) \subseteq H$ is its own normalizer.

4. Suppose that G is a finite group of odd order and H is a subgroup such that $[G : H] = 3$. Show that H is normal in G.

5. Let $G = H \times K$, a direct product of the non-trivial normal subgroups H and K. Let N be any non-trivial normal subgroup of G that intersects both H and K trivially. Show that $N \subseteq Z(G)$.

6. Let G be a finite Abelian group. If n divides the order of G, show that G contains a subgroup of index n.

Rings and Modules: (All rings have an identity element $1 \neq 0$, and all modules are unitary.)

1. Show that R is a division ring if and only if it has no proper non-zero left ideals.

2. Let e be a central idempotent ($e^2 = e$ and e is in the center) of R. Show that

   (a) $1 - e$ is a central idempotent.

   (b) Show that $R \cong eR \times (1 - e)R$ as rings.

3. Suppose that M_1, M_2 and N are submodules of an R-module M, where $M_1 \subseteq M_2$. Show that there is an exact sequence:

$$0 \longmapsto (M_2 \cap N)/(M_1 \cap N) \longmapsto M_2/M_1 \longmapsto (M_2 + N)/(M_1 + N) \longmapsto 0.$$

4. Let $0 \to K \xrightarrow{f} M \xrightarrow{g} L \to 0$ be an exact sequence of R-modules. Prove that the sequence *splits* in the sense that there is a homomorphism $i : L \to M$ such that $g \circ i = 1_L$ if and only if $\mathrm{Ker}(g)$ is a direct summand of M.

5. Prove that if $g : M \to P$ is an epimorphism of R-modules and P is projective, then P is isomorphic to a direct summand of M.

6. Let V be a vector space over a field. Let R be the ring of linear transformations $R = \mathrm{Hom}_K(V, V)$. Prove that V is a simple left R-module.

Fields:

1. Prove that if $f(x)$ is a polynomial over a field F, then there is a finite dimensional extension field E of F such that F has a root in E.

2. Find the Galois group of $x^4 - 2$ over $\mathbb{Q}(i)$, and prove your answer.

3. Suppose that $f(x)$ is irreducible in $F[x]$ and assume that K is a Galois extension field of the field F. Show that all irreducible factors of $f(x)$ in $K[x]$ have the same degree.

4. Suppose that K is an extension field of F and $a \in K$ is algebraic. If the degree of the minimal polynomial $m_a(x)$ of a is odd, show that $F(a^2) = F(a)$.

5. Show that if $K = F(u, v)$ with u, v algebraic over F and u separable over F then K is a simple extension of F.

6. If K is a finite dimensional extension of F show that F is finitely generated and algebraic over K.