Ph.D. Comprehensive Exam in Algebra

August 27, 2003 - Frauke Bleher and Norman Johnson

Please solve exactly three problems from each section. Only three problems from each section will be graded.

1. Groups:

(1) Classify all finite groups G of order p^2, where p is a prime.

(2) Let G be a finite p-group and H a normal subgroup of G. If $Z(G)$ denotes the center of G, show that $H \cap Z(G)$ is nontrivial.

(3) Show that any group of order p^2q, for p and q distinct primes is solvable.

(4) Suppose G is a finite group of order n, and p is the smallest prime dividing n. Prove that any subgroup H of G of index p is normal in G.

(5) Prove that there are exactly 3 isomorphism types of groups of order 75. Please describe the 3 isomorphism types.

(6) (a) Give a definition of a solvable group which works for finite and infinite groups.

(b) In case G is a finite group, prove that G is solvable if and only if there is a finite tower of groups

$$1 = N_0 \leq N_1 \leq \cdots \leq N_k = G$$

such that N_i is normal in N_{i+1} and N_{i+1}/N_i is a cyclic group of prime order.

2. Rings and Modules:

(1) Show that every principal ideal domain is a unique factorization domain. Is the converse true? If yes, prove this, if no, find a counterexample.

(2) Let F be a field and $M_n(F)$ the ring of all $n \times n$ matrices over F. Show that $M_n(F)$ is a simple ring.

(3) Let R be the set of rational numbers a/b, where b is odd.

(a) Show that R is an integral domain.

(b) Find the set $U(R)$ of units of R.

(c) Show that $R - U(R)$ is a maximal ideal of R.

(d) Find all prime ideals of R.

(e) Find all ideals of R and show that R is a principal ideal domain.

(4) Suppose R is a commutative ring with 1 and let I and J be ideals in R.

(a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 + I) \otimes (r + J)$ for some $r \in R$.

(b) Prove that there is an R-module isomorphism $R/I \otimes_R R/J \cong R/(I+J)$ mapping $(r+I) \otimes (s+J)$ to $rs + (I+J)$.

(5) Let R be a ring with 1, and let D, L, M and N be unital left R-modules. Suppose that

$$0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N$$

is an exact sequence of R-modules. Prove that the associated sequence

$$0 \longrightarrow \mathrm{Hom}_R(D, L) \xrightarrow{\varphi_*} \mathrm{Hom}_R(D, M) \xrightarrow{\psi_*} \mathrm{Hom}_R(D, N)$$

is an exact sequence of $\mathbb{Z}$-modules.

(6) Let R be a ring with 1, and let P and Q be unital left R-modules. Prove that $P \oplus Q$ is injective if and only if P and Q are injective.

3. Fields

(1) Prove that the multiplicative group of any finite field is cyclic.

(2) Let F be a field of characteristic p and K and extension field of F. If a in K is both separable and purely inseparable over F, show that a is in F.

(3) Show that $x^5 + 5x^3 - 20x + 5$ is not solvable by radicals.

(4) Let K be a finite separable extension of a field F, of prime degree p. Let c be in K such that $K = F(c)$, and let $(c = c_1), c_2, \ldots, c_p$ be the (distinct) roots of the irreducible polynomial of c over F. Prove that if c_2 lies in K, then K/F is Galois and in fact cyclic.

(5) Let t be transcendental over $\mathbb{R}$ and let $f(x) = x^4 - t \in \mathbb{R}(t)[x]$. Let K be the splitting field of $f(x)$ over $\mathbb{R}(t)$, and determine the Galois group $G = \mathrm{Gal}(K/\mathbb{R}(t))$. Describe fully the lattices of subfields of K and of subgroups of G. Please make sure you match up the subgroups H with the fixed fields K^H.

(6) Let F be a field, and let $f(x) \in F[x]$ be of degree n with splitting field K over F. Show that $[K : F]$ divides $n!$.