Ph. D. Comprehensive Exam. (Analysis)

(Juan Gatica and Surjit Khurana)

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Do any 8 problems.

1. Use Fubini's theorem and the relation $\int_0^\infty e^{-tx}dt = \frac{1}{x}$ to prove that $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$.

2. Assuming $a > 1$, use Residue Theorem to evaluate $\int_0^\pi \frac{d\theta}{a+\cos\theta}$.

3. Suppose f(z) is entire and for every $c \in \mathbb{C}$ there is an n (depending on c) such that $f^n(c) = 0$. Prove that f(z) is a polynomial.

4. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and λ another σ-finite measure on $\mathcal{A}$ such that λ is absolutely continuous with respect to μ. Assume $\frac{d\lambda}{d\mu} = f$. Prove that for every measurable function $g : X \to [0, \infty]$, $\int g d\lambda = \int g f d\mu$.

5. Suppose $(X, \mathcal{A}, \mu)$ be a finite measure space and f and a sequence $\{f_n\}$ are in $L^1(\mu)$. Assume that $f_n \to f$ $a.e.[\mu]$ and $\|f_n\|_1 \to \|f\|_1$. Prove that $\|f_n - f\|_1 \to 0$.

6. Find a harmonic function in D(0, 1) whose value on the boundary is x^3+y.

7. Suppose E is a finite dimensional vector space and $\|.\|_1$ and $\|.\|_2$ are two norms on E. Prove that there are positive numbers a and b such that $a\|.\|_1 \le \|.\|_2 \le b\|.\|_1$.

8. Let $f : D(0,1) \to D(0,1)$ be analytic and $f(\frac{1}{4}) = \frac{3}{4}$. Prove that $f'(\frac{1}{4}) \neq \frac{1}{2}$.

9. Let $(X, \mathcal{A}, \mu)$ is a finite measure space and $f : X \to [0, \infty)$ be integrable. Put $c = \int f d\mu$ and assume $c > 0$. Suppose $a > 0$. Prove that

$$\lim_{n\to\infty} \int n \log(1 + (f/n)^a) d\mu = \begin{cases} \infty, & if\ 0 < a < 1 \\ c, & if\ a = 1 \\ 0, & if\ 1 < a < \infty \end{cases}.$$

10. Assume $(X, \mathcal{A}, \mu)$ is a probability space and $f : X \to [0, 1]$ measurable. Let $G = \{(x, t) : 0 \le t \le f(x)\}$. If λ is the Lebesgue measure on [0. 1] and $\mathcal{B}$ is the σ-algebra of Lebesgue-mesurable sets, prove that G is in the σ-algebra $\mathcal{A} \times \mathcal{B}$ and $\int f d\mu = (\mu \times \lambda)(G)$.