Analysis PhD exam in Analysis

Summer 2005 - Aug. 15, 2:30–5:30 pm

P. Jorgensen, L. Wang

Do eight out of eleven. Show all reasoning and all calculations. Your answers are judged on the completeness and clarity of your arguments.

(1) (a) Suppose $f_1, f_2, \cdots \mathbb{R} \longrightarrow [-\infty, \infty]$ are continuous. What conclusions can you make about the following functions:

$$\lim_{n \longrightarrow \infty} f_n,\ \liminf f_n,\ \limsup f_n,\ \inf f_n,\ \sup f_n?$$

(b) Same question as in (a) but with "measurable" in place of "continuous."

(2) Let f: $[0,1] \longrightarrow \mathbb{C}$, and set

$$f_n(x) = \sum_{j=0}^{2^n-1} \left(2^n \int_{j2^{-n}}^{(j+1)2^{-n}} f(y)\ dy \right) \chi_{[j2^{-n},(j+1)2^{-n})(x)}.$$

(a) What can you say about the limit as $n \longrightarrow \infty$ when f is assumed continuous, when f is only measurable, or when $f \in L^2$?

(b) If $f \in L^2$, show that

$$\lim_{n \longrightarrow \infty} \|f - f_n\|_2 = 0.$$

(3) (a) State Fatou's lemma.

(b) Give an example when the $\leqq$ in Fatou's lemma is strict $<$.

(4) Find and prove the limits ($n \longrightarrow \infty$):

$$\int_0^n \left(1 - \frac{x}{n}\right)^n e^{\frac{x}{2}}\ dx, \text{ and } \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x}\ dx.$$

(5) Let $E \subset \mathbb{R}$ be measurable. Set $f(x) = \lambda((-\infty, x] \cap E)$. Notation: λ for Lebesgue measure.

(a) Is f measurable?

(b) Is f continuous?

Give complete answers!

(6) Let f: $[0,1] \longrightarrow \mathbb{C}$, and suppose $f \in L^\infty$, $\|f\|_\infty > 0$. Set $a_n = \int_0^1 |f|^n \; dx$.

Prove that

$$\lim_{n \longrightarrow \infty} \frac{a_{n+1}}{a_n} = \|f\|_\infty .$$

(7) Let H be a Hilbert space and P: $H \longrightarrow H$ a linear operator satisfying $P^2 = P$.

(a) Show that the following are equivalent:

(i) $P = P^*$

(ii) $\|P\| = 1$.

(iii) $PH \perp (I - P) H$.

(b) Give an example when $P^2 = P$ but $P \neq P^*$.

Note: P^* is the adjoint of P.

(8) Let C denote the continuous functions on $[0,1]$, and set

$$K = \left\{ f \in C \middle| \int_0^{\frac{1}{2}} f(t)\ dt - \int_{\frac{1}{2}}^1 f(t)\ dt = 1 \right\}.$$

Is there an $f \in C$ such that $\|f\| = \inf \{ \|g\| \,| g \in K \}$?

Note: $\|g\| := \max_{t \in [0,1]} |g(t)|$.

(9) Illustrate with an example why

$$(l^{\infty})^* \neq l^1.$$

(10) (a) Give an example of a function which is harmonic but not analytic.

(b) Give an example of a function which is analytic but not entire.

(c) Give an example of a non-constant function which is bounded analytic on $\{z \in \mathbb{C} \mid |z| < 1\}$, one on $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$, $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0,\ \operatorname{Im} z > 0\}$. Is there one on all of $\mathbb{C}$?

(11) Consider

$$f(z) = \operatorname{Im} \left\{ \left(\frac{1+z}{1-z} \right)^2 \right\} \text{ for } z \in D = \{ |z| < 1 \}.$$

Does Poisson's integral representation hold for this function, i.e., is there a measure μ on ∂D such that

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Re} \left\{ \frac{e^{it} + z}{e^{it} - z} \right\} d\mu(t)\,?$$