PhD Comprehensive

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1. Suppose μ is a positive measure on X, $f\colon X \longrightarrow [0,\infty]$ is measurable, $\int x\ f\ d\mu = c$, where $0 < c < \infty$, and α is a constant. Prove that

$$\lim_{n\longrightarrow\infty} \int_X n\ \log\left[1 + (f/n)^{\alpha}\right]\ d\mu = \begin{cases} \infty & \text{if } 0 < \alpha < 1, \\ c & \text{if } \alpha = 1, \\ 0 & \text{if } 1 < \alpha < \infty. \end{cases}$$

Hint: If $\alpha \geq 1$, the integrands are dominated by αf. If $\alpha < 1$, Fatou's lemma can be applied.

2. Let X be a metric space, with metric ρ. For any nonempty $E \subset X$, define

$$\rho_E(x) = \inf\{\rho(x,y) : y \in E\}.$$

Show that ρ_E is a uniformly continuous function on X. If A and B are disjoint nonempty closed subsets of X, examine the relevance of the function

$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

to Urysohn's lemma.

3. Construct a sequence of continuous functions f_n on $[0,1]$ such that $0 \leq f_n \leq 1$ and such that

$$\lim_{n\longrightarrow\infty} \int_0^1 f_n(x)\ dx = 0,$$

but such that the sequence $\{f_n(x)\}$ converges for no $x \in [0,1]$.

4. Show that every compact subset of R^1 is the support of a Borel measure.

5. Suppose V is open in R^k and μ is a finite positive Borel measure on R^k. Is the function that sends x to $\mu(V + x)$ necessarily continuous? lower semicontinuous? upper semicontinuous?

6. Suppose μ is a positive measure on X and $f\colon X \longrightarrow (0, \infty)$ satisfies $\int_X f \, d\mu = 1$. Prove, for every $E \subset X$ with $0 < \mu(E) < \infty$, that

$$\int_E (\log f) \; d\mu \leq \mu(E) \log \frac{1}{\mu(E)}$$

and, when $0 < p < 1$,

$$\int_E f^p \; d\mu \leq \mu(E)^{1-p}.$$

7. Define $u_s(t) = e^{ist}$ for all $s \in R^1$, $t \in R^1$. Let X be the complex vector space consisting of all finite linear combinations of these functions u_s. If $f \in X$ and $g \in X$, show that

$$(f, g) = \lim_{A \longrightarrow \infty} \frac{1}{2A} \int_{-A}^{A} f(t) \overline{g(t)} \; dt$$

exists. Show that this inner product makes X into a unitary space whose completion is a non-separable Hilbert space H. Show also that $\{u_s : s \in R^1\}$ is a maximal orthonormal set in H.

8. Let C be the space of all continuous functions on $[0,1]$, with the supremum norm. Let M consist of all $f \in C$ for which

$$\int_0^{1/2} f(t)\ dt - \int_{1/2}^{1} f(t)\ dt = 1.$$

Prove that M is a closed convex subset of C which contains no element of minimal norm.

9. Find

$$\lim_{A \longrightarrow \infty} \int_{-A}^{A} \frac{\sin \lambda t}{t} e^{itx}\ dt \qquad (-\infty < x < \infty)$$

where λ is a positive constant.