Logic Comprehensive Exam

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The following exam has three parts. One should spend about one hour on each part.

Part I. Work problem 1.

Part II. Work three problems from 2-6.

Part III. Work three problems from 7-11.

Part I

1. State and sketch a proof of the completeness theorem for countable first order languages with equality paying particular attention to how the equality symbol and the homomorphism theorem are used.

Part II

2. A structure A for a first order language L has the **finite model property** iff any sentence σ true in A is true in some finite model F of L. Show the class of all models of L with the finite model property is an $EC\Delta$, i.e., there is a set of sentences Σ of L such that this class of models with the finite model property equals $Mod(\Sigma)$.

3. Recall that two structures are said to agree on the parameters in a formula or term if they assign the same thing to each parameter in that formula or term where the parameters are function symbols f, constant symbols c,relation symbols R, and $\forall$. We always assume equality is in the language L. Prove **carefully** that for all structures A, B that if A and B agree on all the parameters in φ and $s : V \to |A| \cap |B|$, then $A \models \varphi[s]$ iff $B \models \varphi[s]$.

4. Let Σ_1 and Σ_2 be two sets of sentences in a first order language L such that no structure for L is a model of both Σ_1 and Σ_2, prove that there is some sentence τ of L such that $Mod(\Sigma_1) \subseteq Mod(\tau)$ and $Mod(\Sigma_2) \subseteq Mod(\neg\tau)$.

5. (a) State and prove the Generalization Theorem for the theory of deductions given in Enderton.

(b) Show for every φ that $\vdash \exists y \forall x \varphi \to \forall x \exists y \varphi$ where φ is a formula in an arbitrary first order language L.

6. Let $T = Th(< Q, +, \cdot, <, 0, 1 >)$ be the theory of the rational numbers Q with the usual operations and constants in its first order language with equality. Prove that there is a countable model of T which is not isomorphic to $< Q, +, \cdot, < , 0, 1 >$..

Part III

7. Prove the following **Equality Theorem** for the calculus of deductions given in Enderton. If φ is any formula and φ' denotes the result of replacing zero or more free occurrences of x in φ by y where for every occurrence of x which is replaced by y one has that y is substitutable for that free occurrence of x, then $\vdash x = y \rightarrow (\varphi \rightarrow \varphi')$.

8. State and prove the Fixed Point Lemma for A_E assuming the result of question 7 here and any other results you need from the catalog of results in Enderton.

9. State and prove the **Compactness Theorem** for first order languages with equality using the method of ultraproducts of models (here you are not able to use the completeness theorem in your proof but you may assume the Los' property and the existence of ultrafilter theorems).

10. Let $\mathfrak{A}$ be any model of the theory of A_E in the language of A_E as described in Enderton. Prove that $\#Th(\mathfrak{A})$ is not definable in $\mathfrak{A}$ (make sure you recall relevant definitions and results in your argument).

11. Prove that there is no recursively enumerable set of natural numbers A such that $\{[[a]]_1 : a \in A\}$ is equal to the class of all total recursive functions of one variable on the natural numbers.