Logic Comprehensive Exam
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The following exam has three parts. One should spend about one hour on each part.

Part I. Work problem 1.

Part II. Work three problems from 2-6.

Part III. Work three problems from 7-11.

Part I

1. State and sketch a proof of the completeness theorem for countable first order languages with equality paying particular attention to how the equality symbol and the homomorphism theorem are used.

Part II

2. Prove that any two elementarily equivalent finite structures in a first order language with a binary function symbol f and unary relation symbol R as well as with equality symbol $=$ are isomorphic.

3. Show that for any wff. φ of a first order language L and any structure $\mathfrak{A}$ for L that if s and s' are "sequences" in $\mathfrak{A}$ which agree on the free variables of φ, then $\mathfrak{A} \models \sigma[s]$ iff $\mathfrak{A} \models \sigma[s']$.

4. Let T_1 and T_2 be two consistent first order theories in a language L. Given that for any structure $\mathfrak{A}$ of L, $\mathfrak{A}$ is a model of T_1 iff $\mathfrak{A}$ is not a model of T_2. Prove that there are finite sets $F_i \subseteq T_i$ for $i = 1, 2$ such that F_i is equivalent to T_i for $i = 1, 2$.

5. (a) State and prove the Generalization Theorem for the theory of deductions given in Enderton.

(b) State and prove the Deduction Theorem of the theory of deductions given in Enderton.

6. Let T be a first order theory in a language L and let $\sum$ be a set of sentences of L. Suppose for each model $\mathfrak{A}$ of T there is a sentence $\sigma \in \sum$ which is true in $\mathfrak{A}$. Show that there are finitely many sentences $\sigma_1, ..., \sigma_n \in \sum$ such that $\sigma_1 \vee ... \vee \sigma_n$ is a theorem of T.

Part III

7. Prove the following Equality Theorem for the calculus of deductions given in Enderton. If φ is any formula and φ' denotes the result of replacing zero or more free occurrences of x in φ by y where for every occurrence of x which is replaced by y one has that y is substitutable for that free occurrence of x, then $\vdash x = y \rightarrow (\varphi \rightarrow \varphi')$.

8. State and prove the Fixed Point Lemma for A_E assuming the result of question 7 here and any other results you need from the catalog of results in Enderton.

9. Let $\mathfrak{A}$ and $\mathfrak{B}$ be any two infinite linear orderings. Prove that they have exactly the same true universal sentences in the language of linear orderings.

10. Let $\mathfrak{A}$ be any model of the theory of A_E in the language of A_E as described in Enderton. Prove that $\#Th(\mathfrak{A})$ is not definable in $\mathfrak{A}$ (make sure you recall relevant definitions and results in your argument).

11. Let $T = Th(< \omega, +, \cdot, S, <, 0, 1 >)$ be the theory of the natural numbers ω with the usual operations, order $<$, and constants in its first order language with equality. Prove that there is a countable model of T which is not isomorphic to $< \omega, +, \cdot, S, <, 0, 1 >$.