Ph.D. qual. exam. and M.S. comp. exam. on Differential Equations (ODEs). Wednesday August 23, 2006.

Answer 5 out of 9 questions. Show your calculations and justify your answers.

1. Consider the autonomous differential equation

$$\dot{x} = -2x + 1.$$

   Give its corresponding flow map $\varphi_t(x_0)$ at $t = 2$.

2. Consider the autonomous system of differential equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -x_1 + x_2 \\ x_1 x_2 - 1 \end{pmatrix}.$$

   (a) Find all fixed points.

   (b) If a fixed point is hyperbolic say if it is a source, a sink, or a saddle.

3. The origin $(0,0)^T$ is a fixed point of the autonomous system of differential equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -x_2 - x_1^3 \\ x_1 - x_2^3 \end{pmatrix}.$$

   (a) Is the origin $(0,0)^T$ a hyperbolic fixed point?

   (b) Is the function $L(x_1, x_2) := x_1^2 + x_2^2$ a (strict) Lyapunov function at the origin $(0,0)^T$ for this system of differential equations?

   (c) Is the constant solution $x(t) = (0,0)^T$ asymptotically Lyapunov-stable?

4. Consider the flow φ_t of the autonomous system of differential equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_2 x_3 - x_1(x_2 - x_3 - 1) \\ x_1 x_3 - x_2(x_3 - x_1 + 1) \\ x_1 x_2 - x_3(x_1 - x_2 - 1) \end{pmatrix}$$

   and the set of initial values

$$S_0 := \{x = (x_{01}, x_{02}, x_{03})^T \in \mathbb{R}^3 \mid 0 \le x_{0i} \le 1 \text{ for } i = 1, 2, 3\}$$

which is a unit cube of volume $\mathrm{Vol}(S_0) = 1$. What is the volume of the set $\varphi_t(S_0)$ at $t = 1$, i.e., mathematically speaking what is the value of $\mathrm{Vol}(\varphi_1(S_0))$?

5. Consider the nonautonomous system of differential equations

$$\dot{x}_1 = -2tx_2 + t^2 \frac{x_1}{(x_1^2 + x_2^2)^{3/4}}, \quad \dot{x}_2 = 2tx_1 + t^2 \frac{x_2}{(x_1^2 + x_2^2)^{3/4}}$$

in Cartesian coordinates (x_1, x_2) with initial conditions $x_1(0) = 1/\sqrt{2}$, $x_2(0) = 1/\sqrt{2}$ at $t_0 = 0$.

(a) Convert the system of differential equations to polar coordinates (r, θ). Hint: remember that these coordinates are related by $x_1 = r\cos(\theta), x_2 = r\sin(\theta)$ and they satisfy the relations

$$r\dot{r} = x_1\dot{x}_1 + x_2\dot{x}_2, \qquad r^2\dot{\theta} = x_1\dot{x}_2 - \dot{x}_1 x_2.$$

(b) Solve the system of differential equations that you have just obtained in polar coordinates (taking into account the initial conditions $x_1(0) = 1/\sqrt{2}, x_2(0) = 1/\sqrt{2}$).

(c) Express the solution in Cartesian coordinates.

6. Consider the autonomous system of differential equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2x_1x_2 + x_2 \\ x_1^2 - 3x_2^2 + x_1 \end{pmatrix}.$$

Does this system have a periodic solution? (note: a fixed point is not considered as being a periodic solution.)

7. Consider the autonomous system of differential equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_1(2 - x_1 - x_2) \\ x_2(4x_1 - x_1^2 - 3) \end{pmatrix}.$$

Does this system have a periodic orbit or not contained in the region

$$D = \{x \in \mathbb{R}^2 \mid x_2 > 0\}?$$

Hint: Apply Dulac's criterion with $\Psi(x_1, x_2) := 1/(x_1x_2)$.

8. For the following one-dimensional autonomous nonlinear ODE, find all the fixed points as a function of the parameter $\mu \in \mathbb{R}$, determine their Lyapunov-stability, sketch a bifurcation diagram, and find the bifurcation value(s) of the parameter $\mu \in \mathbb{R}$:

$$\dot{x} = x^3 + x^2 - \mu x - \mu = (x+1)(x^2 - \mu)$$

9. Consider the system of ODEs $dy/dt = f(t, y)$ and the following explicit Runge-Kutta method

$$\begin{aligned} Y_1 &= y_0 \\ Y_2 &= y_0 + h\frac{1}{2}f\left(t_0, Y_1\right) \\ Y_3 &= y_0 + h\left(-f\left(t_0, Y_1\right) + 2f\left(t_0 + h/2, Y_2\right)\right) \\ y_1 &= y_0 + h\left(\frac{1}{6}f\left(t_0, Y_1\right) + \frac{2}{3}f\left(t_0 + h/2, Y_2\right) + \frac{1}{6}f\left(t_0 + h, Y_3\right)\right) \end{aligned}$$

   (a) What is the local order of this method?

   (b) What is the stability function $R(z)$ of this method ($z := h\lambda$ and $y' = \lambda y$)?