Ph. D Comprehensive Exam, PDE, Fall, 2005

Choose any SIX out of the nine problems.

Notation: $B_R = \{x \in R^n \mid |x| < R\}$ for $R > 0$.

1. Suppose that $u \in C^2(U) \cap C(\bar{U})$, $U \subset R^n$ bounded, satisfies

$$\begin{cases} Lu = -\Sigma_{i,j=1}^n a_{ij}(x) u_{x_i x_j} + b_i(x) u_{x_i} & = f \;\; in \;\; U \\ u & = g \;\; on \;\; \partial U \end{cases}$$

where $a_{ij}, b_i,\ i,j = 1, ..., n$, are smooth on $\bar{U}$ and L is a uniform elliptic operator, and suppose that $v \in C^2(U) \cap C(\bar{U})$ satisfies

$$\begin{cases} Lv & \geq 1 \;\; in \;\; U \\ v & \geq 0 \;\; on \;\; \partial U. \end{cases}$$

Show that in U

$$inf(-f_-)v(x) + inf(g) \leq u(x) \leq sup(f_+)v(x) + sup(g).$$

2. Show that $u = 1$ is the only smooth solution of the Hamilton-Jacobi PDE problem

$$\begin{cases} u_t + |Du|^2 & = 0 \;\; in \;\; R^n \times (0, +\infty) \\ u & = 1 \;\; on \;\; R^n \times \{t = 0\}. \end{cases}$$

3. Find the entropy weak solution for the nonlinear conservation law

$$u_t + 4uu_x = 0, \;\; x \in R, \; t > 0,$$

with initial data $u(x,0) = \begin{cases} 0 & x \leq 0 \\ 1 & 0 < x \leq 1 \\ 0 & x \geq 1. \end{cases}$

4. (i) Define the Sobolev spaces $W^{1,p}(B_1)$ for $1 \leq p \leq \infty$. Define the trace of u for $u \in W^{1,p}(B_1)$. Define the Hölder spaces $C^{0,\gamma}(B_1)$ for $0 < \gamma < 1$.

(ii) Under what condition can $W^{1,p}(B_1)$ be embedded to $L^q(B_1)$?

(iii) Under what condition can $W^{1,p}(B_1)$ be embedded to $C^{0,\gamma}(B_1)$?

5. When $p = n$, can $W^{1,p}(B_1)$ be embedded to $L^\infty(B_1)$? Why?

6. Let u solve

$$\begin{cases} u_{tt} - \Delta u & = 0 \ \ in \ \ R^3 \times (0, +\infty) \\ u = g, \ u_t & = h \ \ on \ \ R^3 \times \{t = 0\} \end{cases}$$

where g and h are smooth and have compact support.

Show that there is a constant C such that

$$|u(x,t)| \leq \frac{C}{t}, \ \ x \in R^3, \ t > 0.$$

7. Let $g \in L^2(B_1)$.

(i) Define weak solution of

$$\begin{cases} u_t - \Delta u & = 0 \ \ in \ \ B_1 \times (0, +\infty) \\ u & = 0 \ \ on \ \ \partial B_1 \times (0, +\infty) \\ u & = g \ \ on \ \ B_1 \times \{0\}. \end{cases}$$

(ii) Show that

$$\max_{0 \leq t \leq T} \int_{B_1} u^2 dx + \int_{B_1 \times (0,T]} |Du|^2 dx dt \leq C \int_{B_1} g^2 dx$$

for any $T > 0$ and for some universal constant C.

8. Show that there is a unique solution of

$$\begin{cases} -\Delta u & = \cos u \ \ in \ \ B_1 \\ u & = 0 \ \ on \ \ \partial B_1. \end{cases}$$

9. Let $B_R^+ = \{x \in B_R \mid x_n > 0\}$. Assume $u \in W^{1,2}\left(B_R^+\right)$ and u is a weak solution of the equation $\Delta u = 0$ in B_R^+ and the trace of u on $\{x_n = 0\}$ is zero. Prove that $u \in C^\infty\left(\overline{B}_{R/2}^+\right)$ by proving that $\tilde{u}$ defined by

$$\tilde{u}(x_1, ..., x_n) = \begin{cases} u(x_1, ..., x_n) \ \text{for} \ x_n > 0 \\ -u(x_1, ..., -x_n) \ \text{for} \ x_n < 0 \end{cases}$$

belongs to $W^{1,2}(B_R)$ and is a weak solution of $\Delta u = 0$ in B_R as well.