**Ph. D Comprehensive Exam, PDE, Fall, 2006**

Choose any SIX out of the nine problems.

Notation: $B_r = \{x \in R^n \mid |x| < r\}$ for $r > 0$.

1. Let $\Omega$ be a bounded smooth domain in $R^n$ and $L$ a uniform elliptic operator of form

$Lu = -\Sigma_{i,j=1}^n (a_{ij}(x)u_{x_i})_{x_j} + c(x)u,$

where the coefficients $a_{ij}, c$ are smooth functions and $c \geq 0$ on $\bar{\Omega}$.

(i) For $f \in L^2(\Omega)$ define the $H_0^1(\Omega)$ weak solution of the following Dirichlet problem

$$\begin{cases} Lu &= f \ \text{ in } \ \Omega \\ u &= 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

(ii) Show that, if $u$ is a weak solution, then the energy estimate

$\|u\|_{H^1} \leq C\|f\|_{L^2}$

holds, for some constant $C$.

2. Show that $u = 2$ is the only smooth solution of the Hamilton-Jacobi PDE problem

$$\begin{cases} u_t + |Du|^2 &= 0 \ \text{ in } \ R^n \times (0, +\infty) \\ u &= 2 \ \text{ on } \ R^n \times \{t = 0\}. \end{cases}$$

3. Find the entropy weak solution for the nonlinear conservation law

$u_t + 2uu_x = 0, \ \ x \in R, \ t > 0,$

with initial data $u(x,0) = \begin{cases} 0 & x \leq 0 \\ 1 & 0 < x \leq 1 \\ 0 & x \geq 1. \end{cases}$

4. Let $\Omega_T = \Omega \times (0, T]$ where $\Omega \subset R^n$ is open and bounded and $T > 0$ is a constant.

Find a constant $C$ such that for every function $u \in C^2(\Omega_T) \cap C(\bar{\Omega}_T)$ with

$$\begin{cases} u_t - \Delta u &\leq 1 \ \text{ in } \ \Omega_T \\ u &\leq 1 \ \text{ on } \ \Gamma_T, \end{cases}$$

where $\Gamma_T = \bar{\Omega}_T - \Omega_T$, we have $u \le C$ in $\bar{\Omega}_T$.

5. Define the Sobolev spaces $W^{1,p}(B_1)$ for $1 \le p \le \infty$. Under what conditions on $p, q, n$ can $W^{1,p}(B_1)$ be embedded to $L^q(B_1)$?

When $p = n$, can $W^{1,p}(B_1)$ be embedded to $L^\infty(B_1)$? Why?

6. Assume $u \in C^2(R^3 \times R)$ is a solution of $u_{tt} = \Delta u$. Also assume $u(x,0) = 0$ and $u_t(x,0) = 0$ for $|x| \ge 1$. Give an argument that for $t > 1$ we have $u(0,t) = 0$. Is the same true if $u \in C^2(R^2 \times R)$?

7. Let $U \subset R^n$ be open, bounded, connected and with smooth $\partial U$. A function $u \in H^1(U)$ is a weak solution of Neumann's problem

$$\begin{cases} -\Delta u & = f \;\; in \;\; U \\ \frac{\partial u}{\partial \nu} & = 0 \;\; on \;\; \partial U \end{cases}$$

provided

$$\int_U Du \cdot Dv dx = \int_U f v dx$$

for all $v \in H^1(U)$. Given $f \in L^2(U)$, prove that there exists a weak solution of the above problem if and only if

$$\int_U f dx = 0.$$

8. Show that there is a unique solution of

$$\begin{cases} -\Delta u & = \frac{u^2}{u^2+1} \;\; in \;\; B_1 \\ u & = 1 \;\; on \;\; \partial B_1. \end{cases}$$

9. Let $B_r^+ = \{x \in B_r \mid x_n > 0\}$. Assume $u \in W^{1,2}(B_r^+)$ and $u$ is a weak solution of the equation $\Delta u = 0$ in $B_r^+$ and the trace of $u$ on $\{x_n = 0\}$ is zero. Prove that $u \in C^\infty\left(\overline{B}_{r/2}^+\right)$ by proving that $\tilde{u}$ defined by

$$\tilde{u}(x_1, ..., x_n) = \begin{cases} u(x_1, ..., x_n) & \text{for } x_n > 0 \\ -u(x_1, ..., -x_n) & \text{for } x_n < 0 \end{cases}$$

belongs to $W^{1,2}(B_r)$ and is a weak solution of $\Delta u = 0$ in $B_r$ as well.