Comprehensive Exam on PDE, Spring 05

Do five out of the following six problems.

1. Consider the equation
$$u_t + u^3 \cdot u_x = 0$$
on $x, t \in \mathsf{R}, t \geq 0$. Prove that there is a smooth solution for all $t \geq 0$ for the initial value $u(x,0) = \arctan(x)$.

2. Show that there is a constant $C_1 < \infty$ such that for all $u \in W^{1,1}((0,1))$ we have
$$\|u\|_{L^\infty((0,1))} \leq C_1 \|u\|_{W^{1,1}((0,1))}.$$
How does that relate to the Sobolev imbedding theorem ? Is there a constant C_2 such that for all $u \in W^{2,1}((0,1) \times (0,1))$ we have
$$\|u\|_{L^\infty((0,1)\times(0,1))} \leq C_2 \|u\|_{W^{2,1}((0,1)\times(0,1))} \ ?$$

3. Let $\Omega = \{x = (x_1, x_2) : x_1^2 + x_2^2 < 1\}$. Show that there is a constant C such that for all $u \in \left\{ u \in C^2\left(\overline{\Omega}\right) \mid u(x) = 0 \text{ for } x \in \partial\Omega \right\}$ and $x \in \overline{\Omega}$ we have
$$|u(x)| \leq C \max_{x \in \overline{\Omega}} |\Delta u(x)|,$$
but no constant C such that
$$|u(x)| \leq C |\Delta u(x)|$$
for all $u \in \left\{ u \in C^2\left(\overline{\Omega}\right) \mid u(x) = 0 \text{ for } x \in \partial\Omega \right\}$ and $x \in \overline{\Omega}$.

4. Let $\Omega = \{x = (x_1, x_2) : x_1^2 + x_2^2 < 1\}$. Define the space $W^{1,p}(\Omega)$. What is the supremum of all p for which $f(x_1, x_2) = (x_1^2 + x_2^2)^{1/4}$ belongs to $W^{1,p}(\Omega)$?

5. Assume $u \in C^2(\mathsf{R}^3 \times \mathsf{R})\,(u = u(x,t))$ is a solution of the equation $\Delta u = u_{tt}$. Also assume $u(x,0) = u_t(x,0) = 0$ for $|x| \geq 1$. Give an argument that for $t > 1$ we have $u(0,t) = 0$. Is the same true if $u \in C^2(\mathsf{R}^2 \times \mathsf{R})\,(u = u(x,t))$?

6. Suppose u is a smooth solution of $\Delta u = 0$ in R^n. Also assume $|u(x)| \leq C(1 + |x|)$. Prove that u is a polynomial of degree at most one.