Ph.D. Comprehensive Examination in Topology

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Instructions: **Do eight (8) problems, four from each part.**
Parts A and B correspond roughly to the semesters 22M:201, and 22M200 respectively. However, some problems may require ideas from both semesters, and some problems may go beyond what was covered in the course.

This is a "closed book" examination. You should have NO *books or papers of your own.* Please do your work on the note-pads provided. Number your pages as to which exercise you are working. Use only one side of the paper, so that your work will be more readable.

Please indicate here which eight (8) problems you want to have graded.

Four from **Part A** A1 A2 A3 A4 A5 A6

Four from **Part B** B1 B2 B3 B4 B5 B6

Notation:
$\mathbb{R}^n$ is Euclidian n-space, with the usual topology.
S^n is the n-sphere, that is the set of points

$$S^n = \{(x_1, \dots x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + \dots x_{n+1}^2 = 1\}.$$

D^n is the n-ball, that is the set of points

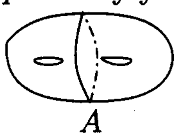
$$D^n = \{(x_1, \dots x_{n+1}) \in \mathbb{R}^n | x_1^2 + \dots x_n^2 \leq 1\}.$$

Note that the boundary of D^n is S^{n-1}.
T^2 is the 2-dimensional torus, $S^1 \times S^1$.
Σ_g is the closed orientable surface of genus g. In particular, $\Sigma_1 = T^2$.
$I = [0, 1]$ is the unit interval in the real numbers.

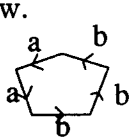
Part A.

A1. For each situation give an example, or explain why no example exists.
(a) A map $f : T^2 \to T^2$ that is a three fold covering map.
(b) A covering map $f : T^2 \to T^2$ so that there are points $x, y \in T^2$ so that the cardinality of $f^{-1}(x)$ is different from the cardinality of $f^{-1}(y)$.
(c) A covering map $f : T^2 \to S^1 \times I$.
(d) A covering map $f : \Sigma_2 \to T^2$.

A2. Let A be a simple closed curve on the closed orientable surface Σ_2 that separates Σ_2 into two punctured tori.
Compute $H_q(\Sigma_2\ ,\ A)$ for all q. Justify your answer.



A3. Let X be the complement of the trefoil knot in the three sphere. It is homotopy equivalent to a CW-complex that has a single 0-cell v, two 1-cells **a** and **b** that are attached to v and a single two cell e which is attached to the 1-skeleton as shown below.



(a) Use the Seifert Van-Kampen theorem to present the fundamental group of X.
(b) Prove that $\pi_1(X, x)$ is not Abelian.

A4. Let X be the space obtained by attaching the disk D^2 to the two sphere S^2 along its equator, so that the attaching map from ∂D^2 to the equator of S^2 has degree 3. Use the Mayer-Vietoris sequence to compute the homology groups of X.

A5. Classify the following list of six spaces up to homotopy type.

1. $\mathbb{R}^2$ - a set consisting of 2 points.
2. S^2 with the north and south poles identified to each other.
3. $S^1 \times S^1$
4. R^3 with the unit circle in the xy-plane removed.
5. R^3 with the unit circle in the xy-plane removed, and the z-axis removed
6. $S^1 \times S^1$ with a point removed.

A6. Give an argument using homology to show that the solid torus $S^1 \times D^2$ does not admit a retraction onto its boundary $S^1 \times S^1$.

Part B.

B1. Let M be the 2-manifold in $\mathbb{R}^3$

$$M = \{(x, y, z) \in \mathbb{R}^3 | z = x^2 + y^2\}$$

The function $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x, y, z) = (2x, 3y, z + 3x^2 + 8y^2)$ restricts to give a smooth map $f : M \to M$.

(a) Find a basis for $T_{(1,2,5)}M$.

(b) Find the image of the basis you chose in (a) under

$$df_{(1,2,5)} : T_{(1,2,5)} \to T_{(2,6,40)}M.$$

B2.

(a) Define the tangent bundle and normal bundle to a manifold $M \subset \mathbb{R}^N$.

(b) What is a vector field on M? What is a non-vanishing vector field?

(c) Does $S^2 \times S^2 \subset \mathbb{R}^6$ have a non-vanishing vector field? Justify your answer.

B3. Can the punctured torus $T^2 - \{\text{one point}\}$ be embedded in the plane $\mathbb{R}^2$? Justify your answer.

B4. For this problem, we assume the following (standard) orientations: Orient $\mathbb{R}^3$ with ordered basis $[i, j, k]$. Orient the xy-plane with $[i, j]$. Give S^2 the boundary orientation from D^3. Orient the z-axis with $[k]$.

Do the following, and explain your reasoning:

(a) Orient the intersection of S^2 with the xy-plane.

(b) Orient the intersection of S^2 with the z-axis.

B5. Suppose that $M^2 \subset \mathbb{R}^5$ is a smooth surface. Prove there is a four-dimensional subspace V of $\mathbb{R}^5$ so that the restriction of the orthogonal projection

$$\pi : \mathbb{R}^5 \to V$$

to M^2 is an immersion.

B6. Let F be a connected two-manifold lying in the three-manifold M.
(a) Prove that each $x \in F$ has a neighborhood $U \subset M$ so that $U - F$ has exactly two path components.
(b) Prove that any other $y \in F$ has an neighborhood $V \subset M$ so that any point in $V - F$ can be connected to a point in $U - F$ by a path in $M - F$.