Ph.D. comprehensive examination in topology
Charles Frohman and Richard Randell
August 23, 2002

Instructions. Do eight problems, four from each part. Parts A and B correspond roughly to the semester courses 22M:201 and 22M:200 respectively. However, some problems may require ideas from both semesters, and some problems may go beyond what was covered in the course. This is a closed book examination. You should have no books or papers of your own. Please do your work on the paper provided. Clearly number your pages to correspond with the problem you are working. You may use "big theorems" (i.e. Mayer-Vietoris, Sard) provided that the point of the problem is not the proof of the theorem. Always justify your answers.

Please indicate here which eight problems you want to have graded:

A1 A2 A3 A4 A5 A6 B1 B2 B3 B4 B5 B6

Notation: R^n is Euclidean n-space, with the usual topology and differentiable structure.

S^n is the n-sphere, the set of points distance one from the origin in R^{n+1}, with the subspace topology, and with the usual differentiable structure.

D^n is the closed unit ball, the set of points distance one or less from the origin in R^{n+1}, with the subspace topology and the usual differentiable structure.

$I = [0, 1]$ is the closed unit interval in the real numbers.

The n-torus T^n is the cartesian product of n copies of S^1.

"Manifold" means compact differentiable manifold without boundary, unless otherwise noted

Part A

A1) Use a Mayer-Vietoris sequence argument to compute the homology groups of the 3-torus T^3.

A2) Let $A \subset S^2$ be defined by $A = \{(x, y, z) \in R^3 \mid z = 0 \text{ or } z = \pm 1\}$. Compute $H_*(S^2, A)$. Justify your answer.

A3) Give an example of a space with non-abelian fundamental group. Prove that the group is not abelian.

A4) Give a surjective homomorphism from the fundamental group of a space to the first homology group of the space. Prove that this homomorphism is natural with respect to continuous functions, and is surjective. (You may or may not need to impose additional hypothesis on the spaces involved.)

A5) Let $p : E \to B$ be a covering map, with E and B path-connected. Prove or give counterexample to:

a) $p_* : H_2(E) \to H_2(B)$ is $1-1$.
b) $p_* : H_2(E) \to H_2(B)$ is onto.

A6) Let $h : I \to S^n$ be an embedding. (h is a homeomorphism onto its image). Prove that $H_i(S^n - h(I)) = 0$, for $i > 0$. (Here of course $I = [0, 1]$ is the unit interval).

Part B

B1) Let M be a compact smooth manifold and let $f : M \to R$ be smooth. Prove that either f has at least two critical values, or f is a constant map.

B2) Does there exist a smooth embedding of the Möbius band into the plane R^2? Justify your answer.

B3) Let $SO(n)$ be the set of all $n \times n$ matrices with real entries and determinant 1, with the property that $AA^t = Id$. Prove that $SO(n)$ is a smooth manifold and give its dimension.

B4) Let RP^2 denote the projective plane and let $A \subset RP^2$ be defined by $A = \{[x : y : z] \mid x^2+y^2-z^2 = 0\}$ (in homogeneous coordinates for RP^2.). Prove or disprove: A is a submanifold.

B5) Let T be the subset of R^3 obtained by rotating the circle of radius 3 centered at $(5,0)$ in the xy-plane around the y-axis. Let $f : T \to R^2$ be projection to the yz-plane, and let $g : T \to R^1$ be projection to the y-axis.

a) Find the critical points of f by a geometric analysis.

b) Find the critical points of g by a geometric analysis.

c) Let $\theta : T \to R^3$ be given by $\theta(p) = (f(p), g(p))$. Is θ an immersion? an embedding? Justify geometrically.

B6) Let M be a smooth manifold with non-empty boundary ∂M. Prove that ∂M is a smooth manifold without boundary