Ph.D. comprehensive examination in topology

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August 22, 2003

Instructions. Do eight problems, four from each part. Parts A and B correspond roughly to the semester courses 22M:201 and 22M:200 respectively. However, some problems may require ideas from both semesters, and some problems may go beyond what was covered in the course. This is a closed book examination. You should have no books or papers of your own. Please do your work on the paper provided. Clearly number your pages to correspond with the problem you are working. You may use "big theorems" (i.e. Mayer-Vietoris, Sard) provided that the point of the problem is not the proof of the theorem. Always justify your answers.

Please indicate here which eight problems you want to have graded:

A1 A2 A3 A4 A5 A6 B1 B2 B3 B4 B5 B6

Notation: $\mathbb{R}^n$ is Euclidean n-space, with the usual topology and differentiable structure.

S^n is the n-sphere, the set of points distance one from the origin in $\mathbb{R}^{n+1}$, with the subspace topology, and with the usual differentiable structure. Real projective n-space $\mathbb{R}P^n$ is the quotient $\mathbb{R}^{n+1} - \{0\}$ under the equivalence relation induced by $\overline{v} \cong \overline{w}$ if there is $\lambda \in \mathbb{R}$ so that $\lambda \overline{v} = \overline{w}$. The map $q : S^n \to \mathbb{R}P^n$ given by sending each $\overline{v} \in S^n$ to its equivalence class is a quotient map.

D^n is the closed unit ball, the set of points distance one or less from the origin in $\mathbb{R}^{n+1}$, with the subspace topology and the usual differentiable structure.

$I = [0, 1]$ is the closed unit interval in the real numbers.

The n-torus T^n is the cartesian product of n copies of S^1.

"Manifold" means compact differentiable manifold *without* boundary, unless otherwise noted

1 Part A

A1 Let

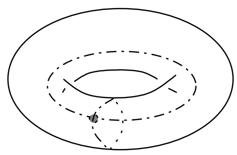
$$0 \to C \to D \to E \to 0$$

be a short exact sequence of chain complexes. Define the connecting homomorphism $\Delta : H_i(E) \to H_{i-1}(C)$ and prove that it is independent of the choices involved.

A2 Let $A \subset T^2$ be defined by

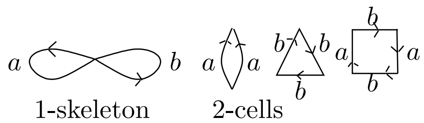
$$A = \{(x, y, z, w) \in T^2 | (x, y) = (0, 1) \text{ or } (z, w) = (0, 1)\}.$$

Compute $H_*(T^2, A)$. *We are thinking of S^1 as the unit circle in the plane, hence we indicate points on S^1 as ordered pairs, below is a picture of the torus with A indicated.*



Justify your answer.

A3 Let X be a CW-complex with one vertex, two one cells and 3 two cells whose attaching maps are indicated below.

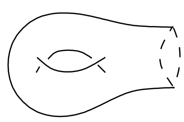


1-skeleton 2-cells

a. Compute the homology of X.
b. Present the fundamental group of X and prove its nonabelian.
Justify your work.

A4

a. Let $\Sigma_{1,1}$ be the orientable surface obtained by removing a closed disk from T^2. Compute the homology of $\Sigma_{1,1}$.



b. Let Σ_2 denote the closed orientable surface of genus 2. Use your answer from part **a** and the Mayer-Vietoris sequence to compute the homology of Σ_2.

A5 a. Give a definition of covering space where the base space is locally path connected.

b. Give careful statements of existence an uniqueness theorems for lifting paths and homotopies.

c. Prove that if $p : X \to B$ is a covering space and B and X are both path connected then the cardinality of $p^{-1}(b)$ is independent of the point $b \in B$ you choose.

A6 Let $h : I \times I \to S^n$ be an embedding. (h is a homeomorphism onto its image). Prove that $H_i(S^n - h(I \times I)) = 0$, for $i > 0$. *You may use the fact that if $g : I \to S^n$ is an embedding then $H_i(S^n - g(I))) = 0$ for $i > 0$. Also, as mentioned on page 1, $I = [0, 1]$.*

2 Part B

B1 Let M be a compact two-manifold embedded in $\mathbb{R}^3$. Prove that almost all vertical lines in $\mathbb{R}^3$ intersect M in a finite set of points (possibly empty).

B2 Let $f : M \to N$ be smooth, where M and N are smooth, compact connected manifolds of the same dimension.

a. Prove that f is a submersion if and only if f is an immersion.

b. Prove or give a counterexample: f an immersion implies f is injective.

c. Prove or give a counterexample: f a submersion implies f is onto.

B3 a. Prove that $Gl(n, \mathbb{R})$, the group of all non-singular $n \times n$ matrices with real entries, is a Lie group.

b. Prove that every Lie group is parallelizable.

B4 Let M be a compact manifold with boundary. Prove that there is no smooth function $f : M \to \partial M$ with $f\,|_M = Id$.

B5 Define what it means for a smooth function $f : M \to N$ to be transverse to a submanifold $L \subset N$. Give proof or counterexample to:

a) f transverse to L implies $f(M) \cap L$ is a submanifold of N.

b) f transverse to L implies $f^{-1}(L)$ is a submanifold of M.

B6 Consider the function $f : S^2 \to S^2$ given by $f(x, y, z) = (-z, -x, y)$. Explain why f does or does not induce a smooth function $g : \mathbb{R}P^2 \to \mathbb{R}P^2$. If the answer is yes, compute the differential dg at the point of $\mathbb{R}P^2$ corresponding to $\frac{1}{\sqrt{3}}(1, 1, 1)$. If the answer is no, tell how to modify f slightly so that it does induce a smooth g.