Ph.D. Comprehensive Examination in Topology

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Instructions. Do eight problems, four from each part. Some problems may require ideas from both semesters, and some problems may go beyond what was covered in the course. This is a closed book examination. You should have no books or papers of your own. Please do your work on the paper provided. Clearly number your pages to correspond with the problem you are working. You may use "big theorems" (i.e. Mayer-Vietoris, Sard) provided that the point of the problem is not the proof of the theorem.

Always justify your answers unless explicitly instructed otherwise.

Please indicate here which eight problems you want to have graded:

A1 A2 A3 A4 A5 A6
B1 B2 B3 B4 B5 B6

Notation:
$\mathbb{R}^n$ is Euclidean n-space, with the usual topology and differentiable structure. S^n

is the n-sphere, the set of points distance one from the origin in $\mathbb{R}^{n+1}$, with the subspace topology, and with the usual differentiable structure.

"Manifold" means compact differentiable manifold without boundary, unless otherwise noted

A "map" is a continuous function.

Part A

A1) A map from S^n to S^n is said to be *orientation-preserving* if the degree of the map is positive.

(a) For which values of n is the antipodal map $\alpha : S^n \to S^n$ given by $\alpha(x_1, \ldots, x_{n+1}) = (-x_1, \ldots, -x_{n+1})$ orientation preserving?

(b) Give an example of an orientation preserving map of $S^2 \to S^2$ that is NOT homotopic to the identity map.

A2) A soccer ball is made (topologically) out of pentagons and hexagons sewn together along edges. At each vertex, there are two hexagons and one pentagon.

(a) How many pentagons and how many hexagons must there be?

(b) Suppose we tried to make a "torus shaped soccer ball" (i.e. a surface that is topologically a torus) out of pentagons and hexagons, still obeying the rule that at each vertex, there are exactly two hexagons and one pentagon. Explain why this is impossible.

A3) Suppose $p : E \to X$ is a covering space, where E is simply connected and B is locally path connected. Let $q : \widetilde{X} \to X$ be some other covering space. Prove there exists a covering map from E to $\widetilde{X}$.

A4) Let X be the subset of $\mathbb{R}^3$ obtained by rotating the circle of radius 1 centered at $(1, 0, 0)$ in the $xy-$plane around the $y-$axis. Calculate the homology groups of X.

A5) Suppose we are in the following situation: We have a space X (that is a cell complex) and we form a new space Y by adjoining a 3-cell B to X via an attaching map $f : bdy\, B \to X^{(2)}$. (Here $X^{(2)}$ denotes the 2-skeleton of X.) Describe how the ranks of the various homology groups of X and Y are related.

A6) Prove

$$\pi_1\left((X \times Y), (x_0, y_0)\right) \cong \pi_1\left(X, x_0\right) \oplus \pi_1\left(Y, y_0\right) .$$

Part B

B1) Give a definition of an n−form on a smooth manifold. Compute the exterior derivative of the one-form $(x + y)dx \wedge yxdy$ on $\mathbb{R}^2$. Give an example of a nowhere-zero 2−form on S^2.

B2) Define the concepts of flow and vector field on a smooth manifold M. Prove that a flow on M induces a vector field on M.

B3) Let $SO(n)$ be the set of all $n \times n$ matrices with real entries and determinant 1, with the property that $AA^t = Id$. Prove that $SO(n)$ is a smooth manifold and give its dimension.

B4) Let $\mathbb{RP}^2$ denote the projective plane and let $A \subset \mathbb{RP}^2$ be defined by $A = \{[x : y : z] \mid x^4 + y^4 - z^4 = 0\}$ (in homogeneous coordinates for $\mathbb{RP}^2$.). Prove or disprove: A is a submanifold.

B5) Let T be the subset of $\mathbb{R}^3$ obtained by rotating the circle of radius 3 centered at $(5, 5, 0)$ in the xy−plane around the x−axis. Let $f : T \to \mathbb{R}^2$ be projection to the yz-plane, and let $g : T \to \mathbb{R}^1$ be projection to the y-axis.
a) Find the critical points of f by a geometric analysis.
b) Find the critical points of g by a geometric analysis.
c) Let $\theta : T \to \mathbb{R}^3$ be given by $\theta(p) = (f(p), g(p))$. Is θ an immersion? Justify geometrically.

B6) Define what it means to say a function $f : X \to Y$ is smooth. Prove that the differential df on tangent spaces is given by the Jacobian of partial derivatives with respect to some choices of bases for the tangent spaces. Describe the bases you use; you do not need to prove that these are bases.