Ph.D. Comprehensive Examination in Topology

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Instructions: **Do eight (8) problems, four from each part.** Parts A and B correspond roughly to the semesters 22M:201, and 22M200 respectively. However, some problems may require ideas from both semesters, and some problems may go beyond what was covered in the course. *This is a "closed book" examination. You should have* NO *books or papers of your own.* Please do your work on the notepads provided. Number your pages as to which exercise you are working. Use only one side of the paper, so that your work will be more readable.

Please indicate here which eight (8) problems you want to have graded.
Problem A1 A2 A3 A4 A5 A6 B1 B2 B3 B4 B5 B6

Notation:
$\mathbb{R}^n$ is Euclidian n-space, with the usual topology.
S^n is the n-sphere, the set of points

$$S^n = \{(x_1, \dots x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + \dots x_{n+1}^2 = 1\}.$$

D^n is the n-ball, that is the set of points

$$D^n = \{(x_1, \dots x_n) \in \mathbb{R}^n | x_1^2 + \dots x_n^2 \leq 1\}.$$

Note that the boundary of D^n is S^{n-1}.
The n-torus T^n is the cartesian product of n copies of S^1.
RP^2 is the projective plane, the quotient of S^2 by the antipodal action.
"Manifold" means compact differentiable manifold without boundary, unless otherwise noted
$I = [0, 1]$ is the unit interval in the real numbers.

Part A.

A1. a. Prove that if $h : [0, 1] \to S^3$ is an embedding then $H_i(S^3 - h([0, 1])) = 0$ if $i > 0$ and $H_0(S^3 - h([0, 1])) = \mathbb{Z}$.
b. If $h : S^1 \to S^3$ is an embedding compute $H_i(S^3 - h(S^1))$.

A2. Use covering spaces to prove that any subgroup of finite index of a free group of finite rank is free.

A3. Classify the following list of spaces according to homotopy type. Justify your answer.

1. The torus $S^1 \times S^1$.
2. The torus $S^1 \times S^1$ with $S^1 \times \{(1, 0)\}$ identified to a point.
3. The path connected CW-complex that looks like the greek letter Θ.
4. The complement of two disjoint lines in $\mathbb{R}^3$.
5. The complement of a point in the two torus $S^1 \times S^1 = T^2$.
6. The wedge of a circle and a sphere, $S^1 \vee S^2$.
7. The sphere S^2 with its north and south poles $(0, 0, 1)$ and $(0, 0, -1)$ identified to a point.

A4. State the Hurewicz isomorphism theorem (i.e. the theorem relating π_1 and H_1). Outline the construction of the homomorphism and the proof of the theorem.

A5. Suppose the CW complex Y is formed by attaching three two-cells to S^1, via maps $f_i : S^1 \to S^1$, of degree 0, 1, and 7, respectively. Determine the homology groups of Y.

A6. a. Give a careful statement of the excision theorem for singular homology.
b. Prove that if (X, A) is a pair so that X is compact hausdorff, A is closed and there is U open with $A \subset U$ so that A is a strong deformation retract of U then $H_*(X, A)$ is isomorphic to the reduced homology $\overline{H}_*(X/A)$ of the quotient space X/A obtained from X by identifying A to a point.

Part B.
B1.
a. Let $O(n)$ be the space of $n \times n$ matrices A with $AA^t = I$. Prove that $O(n)$ is a smooth manifold and compute its dimension. State clearly and carefully and theorems that you use.
b. What is the tangent space of $O(n)$ at the identity matrix?

B2. Does there exist a smooth embedding of the projective plane RP^2 into the plane R^2? Justify your answer.

B3. Define **Lie Group**. Prove that any Lie group G is parallelizable. (That is, prove that the tangent bundle of G is diffeomorphic to $G \times R^n$, where n is the dimension of G.)

B4. Formulate and prove the chain rule for the differential of the composite $d_p(g \circ f)$, in terms of df and dg, for $f : M \to N, g : N \to L$.

B5. Let M be a compact two-manifold (without boundary) embedded in R^3. Prove that almost every plane through the origin intersects M in either the empty set, or in a compact 1-manifold.

B6. Write a short essay describing the relationship between an abstract notion of tangent vector (i.e. not coming from the manifold as a subset of Euclidean space) with the geometric notion of tangent vector to a manifold embedded in Euclidean space. Include a discussion of the differential map df induced by a smooth map f.