

# Ph.D. Comprehensive Examination in Topology

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*Instructions: Do eight (8) problems, four from each part.* Parts A and B correspond roughly to the semesters 22M:201, and 22M200 respectively. However, some problems may require ideas from both semesters, and some problems may go beyond what was covered in the course. *This is a “closed book” examination. You should have NO books or papers of your own.* Please do your work on the notepads provided. Number your pages as to which exercise you are working. Use only one side of the paper, so that your work will be more readable.

Please indicate here which eight (8) problems you want to have graded.

**four from Part A and four from Part B**

A1 A2 A3 A4 A5 A6 B1 B2 B3 B4 B5 B6

*Notation:*

$\mathbb{R}^n$  is Euclidian n-space, with the usual topology.

$S^n$  is the n-sphere, the set of points

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}.$$

$D^n$  is the n-ball, that is the set of points

$$D^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 \leq 1\}.$$

We will denote vectors by using bold font, for example  $\mathbf{i}$ . We use the convention that  $\mathbf{i} = (1, 0) \in \mathbb{R}^2$  or  $(1, 0, 0) \in \mathbb{R}^3$ ;  $\mathbf{j} = (0, 1)$  or  $(0, 1, 0)$ , etc.. The vector  $\mathbf{0}$  is the vector all of whose entries are zero.

Note that the boundary of  $D^n$  is  $S^{n-1}$ .

$T^2$  is the 2-dimensional torus,  $S^1 \times S^1$ .

$I = [0, 1]$  is the unit interval in  $\mathbb{R}^1$ .

**Part A.**

**A1.**

Suppose  $X = A \cup B$ , where  $A \cap B = \{p\}$ , a single point. Use the Mayer-Vietoris sequence to calculate the homology groups of  $X$ , expressed in terms of the homology groups of  $A$  and  $B$ .

**A2.**

Outline a proof that if  $f, g : X \rightarrow Y$  are homotopic maps, then for all  $n$ , the homomorphisms  $f_*, g_* : H_n(X) \rightarrow H_n(Y)$  are equal.

**A3.**

a. Define the relative homology groups  $H_n(X, A)$ .

b. Consider the sphere  $S^n$  as the union of upper and lower hemispheres,  $S^n = D_+^n \cup D_-^n$ , with  $D_+^n \cap D_-^n = S^{n-1}$ .

Use the *excision theorem* to prove (for all  $k$ ),  $H_k(S^n, D_-^n) \cong H_k(D_+^n, S^{n-1})$ .

**A4**

Suppose  $f : D^n \rightarrow D^n$  is a continuous function such that the restriction  $f|_{S^{n-1}} : S^{n-1} \rightarrow S^{n-1}$  is a homeomorphism.

Prove the map  $f : D^n \rightarrow D^n$  is surjective.

Hint: If  $p$  is any point in the interior of  $D^n$ , then the set  $D^n - \{p\}$  deformation-retracts to  $S^{n-1}$ .

**(Part A continued next page.)**

**A5.**

a. State the fundamental group of each of the following spaces. Explain (briefly) how you determine each group.

- $S^1 \times S^1 \times S^2$
- a space  $X = M_1 \cup M_2$ , where  $M_1$  and  $M_2$  are Möbius bands, and  $M_1 \cap M_2 = \text{boundary}(M_1) = \text{boundary}(M_2)$ .
- The set  $X = \mathbb{R}^3 - W$ , where  $W$  is the union of two straight lines that intersect at the origin.

**A6**

Let  $X$  be the following subspace of  $\mathbb{R}^5$ :

$$X = \{(x_1, x_2, x_3, x_4, x_5) \mid x_1^2 + x_2^2 + x_3^2 = 1 \text{ and } x_4^2 + x_5^2 = 1\}$$

- Describe (with equations or words) the universal covering space of  $X$ .
- Describe (with equations or words) a 3-fold covering map  $p : X \rightarrow X$ .

**End of Part A**

**Part B.**

**B1. a.** What does it mean to say that  $X$  and  $Y$  are transverse smooth submanifolds of the manifold  $M$  at the point  $p$ ? What does it mean to say that  $X$  and  $Y$  are transverse?

**b.** If the dimension of  $X$  is  $i$ , the dimension of  $Y$  is  $j$  and the dimension of  $M$  is  $m$ , and  $X$  and  $Y$  are transverse in  $M$ , what can you say about  $X \cap Y$ ?

**c.** For each of the situations below say whether  $X$  and  $Y$  are transverse in  $M$ . Justify your answer.

- Let  $X$  be the  $x$ -axis,  $Y$  be the  $y$ -axis and  $M = \mathbb{R}^3$ .
- Let  $X$  be the  $z$ -axis,  $Y$  be the unit sphere centered at the origin, and  $M = \mathbb{R}^3$ .
- Let  $X$  be the unit sphere centered at the origin,  $Y$  be the  $xy$ -plane and  $M = \mathbb{R}^3$ .
- Let  $X$  be the unit sphere centered at the origin,  $Y$  be the plane  $z = 1$  and  $M = \mathbb{R}^3$ .

**B2.** Let  $M$  be a smooth compact manifold with boundary. Prove that there does not exist a smooth map  $r : M \rightarrow \partial M$  so that its restriction to the boundary is the identity map. Carefully state any theorems you use.

**B3.** Prove that any compact smooth manifold  $M$  of dimension  $n$  can be embedded in  $\mathbb{R}^{2n+1}$ . You may assume that  $M$  is embedded in some  $\mathbb{R}^N$  to begin with where  $N$  is large. Carefully state any theorems you use.

**B4.** Can the Möbius band be embedded in the plane? Give careful statements of any theorems that you use.

**B5.** Let  $O(n)$  denote the set of  $n \times n$  matrices  $A$  with real coefficients, so that  $AA^t$  is the identity map. Prove that  $O(n)$  is a smooth manifold. What is its dimension? Once again you need to give careful statements of any theorems you use.

**B6.** Let  $T$  be the torus obtained by rotating the circle of radius  $1/2$  centered at  $(0, 1, 0)$  and lying in the  $yz$ -plane, about the  $z$ -axis in  $\mathbb{R}^3$ . Let  $S$  be the unit sphere centered at the origin. Let  $\pi : T \rightarrow S$  be the map 
$$\pi(x, y, z) = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}.$$

- a. At what points of  $T$  does  $\pi$  fail to be a submersion?
- b. What are the regular values of  $\pi$ ?