Ph. D. Comprehensive Exam. (Analysis)

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Do any 8 problems.

1. Show that $\sin z$ is entire. Discuss the analyticity of $\sin^{-1} z$. Find the first few terms in the power series expansion of $\sin^{-1} z$ at '0' and determine the circular nbd. in which the series is convergent.

2. Evaluate the $\int_0^\infty \frac{\log x}{(1+x^2)^2} dx$, using the contour integral and residue method.

3. Evaluate the following limits as $n \to \infty$, justifying every step:
(*i*) $\int_0^n (1 - \frac{x}{n})^n e^{\frac{x}{2}} dx$; (*ii*) $\int_0^n (1 + \frac{x}{n})^n e^{-2x} dx$.

4. Evaluate: $\underset{(a,b,c)}{min} \int_0^\infty |x^3 - a - bx - cx^2|^2 e^{-x} dx$.

5. Show that $M = \{f \in L^1(0,1) : \int_0^1 f(t)dt = 1\}$ is closed and and convex in $L^1(0,1)$and it infinity many elements of minimum norm.

6. With Lebesgue measure λ on [0, 1], let $f : [0,1] \to [0,\infty]$ be Lebesgue integrable and $\int f d\lambda = \int f^n d\lambda$ for every positive integer n. Prove that f is a charateristic function.

7. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, $\{f_n\}$ a sequence of measurable functions such that $f_n \to 0$ *a.e.* and $\int |f_n|^2 d\mu \leq 1$, $\forall n$. prove that $\|f_n\|_1 \to 0$.

8. Suppose a seqence $\{x_n\}$ is orthogonal sequence in a Hilbert space H such that $\sum_1^\infty < x_n, y >$ is convergent for each y in H. Does this imply that $\sum_1^\infty x_n$ is convergent in H? Justify your answer.

9. Let $f : \mathbb{C} \to \mathbb{C}$ be entire 1-1, onto. Prove that f(z) is a linear polynomial.

10. Prove that $\int_0^1 \int_1^\infty (e^{-xy} - 2e^{-2xy}) dx dy \neq \int_1^\infty \int_0^1 (e^{-xy} - 2e^{-2xy}) dy dx$. Why they are unequal? Justify your answer.