Ph.D. Comprehensive Exam–PDE Spring, 2001

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Solve any 6 problems. Have fun!

QuestionItem 1. Let Ω be a bounded smooth domain in $R^n, n \geq 1$. Let u be a classical solution of

$$\begin{cases} u_t - \Delta u = f(x) & \text{in} \quad \Omega \times (0,\infty) \\ u = 0 & \text{on} \quad \partial\Omega \times (0,\infty) \\ u = g(x) & \text{on} \quad \Omega \times \{0\}. \end{cases} \qquad \text{(ppde)}$$

Show that

$$\max_{0 \leq t \leq T} \int_\Omega u^2(x,t)dx + \iint_{\Omega\times(0,T)} |\nabla u(x)|^2 dxdt \leq C\left(\int_\Omega g^2(x)dx + \iint_{\Omega\times(0,T)} |f(x)|^2 dxdt \right)$$

for some universal constant C.

2. (a). Assume that $F : R \mapsto R$ is C^1, with F' bounded. Suppose that $\Omega \subset R^n$ is a bounded domain and $p \in (1,\infty)$. Show that if $u \in W^{1,p}(\Omega)$, then $v \equiv F(u) \in W^{1,p}(\Omega)$ and $\frac{\partial v}{\partial x_i} = F'(u)\frac{\partial u}{\partial x_i}$.

(b). Verify that $u(x,y) = xy(\ln r)^\alpha$ with $r \equiv \sqrt{x^2+y^2}$ and $\alpha \in (0,1)$ satisfies $\Delta u(x,y) = f(x,y)$ for some $f \in C(R^2)$ and $u \notin C^2(R^2)$, which shows that the "gain of two" regularity theorem is false in the C^m classes.

3. Use the result in 1 to show the uniqueness of classical solution of (ppde).

4. Let Ω be a bounded smooth domain in $R^n, n \geq 1$ and

$$I[u] \equiv \int_\Omega (\frac{1}{2}|\nabla u(x)|^2 + F(x,u(x)))dx, u \in H_0^1(\Omega),$$

where F is a smooth function and $\frac{\partial F}{\partial u} = f(x,u)$.

(a). What growth condition(s) do we need to impose on F so that I is well-defined?

(b). What is the Euler-Lagrange equations of I?

(c). What condition(s) do we need to impose on F to ensure that I is weakly lower semicontinuous?

(d). What condition(s) do we need to impose on F to ensure the existence of a **unique** minimizer of I?

5. Let Ω be a bounded domain in $R^n, n \geq 1$ and L a uniform elliptic operator of the form

$$Lu = -\sum_{i,j=1}^n (a_{i,j}(x)u_{x_i})_{x_j} + c(x)u,$$

where the coefficients $a_{i,j}$ are smooth (symmetric) functions and $c \geq 0$ on $\overline{\Omega}$.

(a). For $f \in L^2(\Omega)$ define the $H_0^1(\Omega)$ weak solution of the following Dirichlet problem:

$$\begin{cases} Lu = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

(b) Show that, if u is a weak solution then this energy estimate holds $\| u \|_{H^1} \leq C \| f \|_{L^2}$.

6. Show that for any $f \in L^2(\Omega)$ the above equation in #5 has a unique weak solution. Under what condition on f does one have

(a) $u \in H^2$?
(b) $u \in C^2$?
7. Show that there is always a unique solution of

$$\begin{cases} \Delta u = \frac{e^u}{e^u+1} & \text{in} \quad B_1, \\ u = 1 & \text{on} \quad \partial B_1, \end{cases}$$

Is the solution radially symmetric, why?
8. (a). What are the characteristic ODE system for the following Cauchy problem?

$$\begin{cases} u_t + (\frac{u^2}{2})_x = f(x,u) & \text{in} \quad R \times (0,\infty), \\ u(x,0) = h(x) & \text{on} \qquad R. \end{cases}$$

(b). Is the function

$$u(x,t) = \begin{cases} -\frac{2}{3}(t + \sqrt{3x+t^2}\,) \text{ when } 4x + t^2 > 0, \\ 0 \text{ when } 4x + t^2 < 0, \end{cases}$$

a weak solution when $f = 0$?