Ph.D. Comprehensive Examination in Topology

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Instructions: **Do eight (8) problems, four from each part.** Parts A and B correspond roughly to the semesters 22M:201, 22M:200. However, some problems may require ideas from both semesters, and some problems may go beyond what was covered in the course. *This is a "closed book" examination. You should have NO books or papers of your own.*

Please indicate here which eight (8) problems you want to have graded.

Problem	A1	A2	A3	A4	A5	A6	B1	B2	B3	B4	B5	B6

Notations.

R^n is Euclidean n-space, with the usual topology.

S^n is the n-sphere, the n-dimensional unit sphere in R^{n+1}.

$D^n = \{x \in R^n \mid ||x|| \leq 1\}$ is the n-imensional unit disk in R^n.

Part A.

A1. State and prove the Brower fixed point theorem for D^2.

A2. Let $f : \partial D^2 \to S^1$ be a 3-fold covering map, and let $X = D^2 \cup_f S^1$ be the space obtained by gluing the boundary of D^2 to S^1 using the map f.

(a) Calculate the fundamental group and homology groups of X.

(b) Describe the universal covering space of X.

A3. Write down the homology groups of the following spaces, and classify them up to homotopy equivalence.

(a) R^3 with two coordinate axes removed;

(b) $S^2 \vee S^1$;

(c) RP^2 with two points removed;

(d) $S^1 \times S^1$ with one point removed.

A4. A short exact sequence of abelian groups $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is said to be *split* if there is a homomorphism $r : C \to B$ such that $jr = id_C$. Show that

(a) if C is a free abelian group, then the sequence is split;

(b) if the sequence is split, then $\exists$ a homomorphism $t : B \to A$ such that $ti = id_A$.

A5. Let $A \neq \emptyset$ be a path connected subset of X, and $j : A \to X$ the inclusion map.

(a) Show that if $j_* : \tilde{H}_*(A) \to \tilde{H}_*(X)$ is a trivial map, and $H_*(A)$ is a free abelian group, then $H_q(X, A) \cong \tilde{H}_q(X) \oplus \tilde{H}_{q-1}(A)$ for all q.

(b) Find an example of (X, A) such that $H_q(X, A) \ncong \tilde{H}_q(X) \oplus \tilde{H}_{q-1}(A)$.

A6. (a) Show that if M is an n-dimensional manifold (without boundary), then

$$H_n(M, M - p) = Z \text{ for all } p \in M.$$

(b) Let $X = S^2 \vee S^2$. Calculate $H_2(X, X - p)$, and show that X is not a 2-manifold.

Part B.

B1. Here is a way to construct some 2-manifolds with boundary. Let X^2 and Y^1 be compact manifolds without boundary in R^3. Suppose X^2 is transversal to Y^1. At each point $x \in X \cap Y$, remove a small open disk neighborhood $U(x)$ from X. Let $W = X - \bigcup_{x \in X \cap Y} U(x)$.

(a) Explain how we know that we can pick these disk neighborhoods $\{U(x)\}$ to be pairwise disjoint.

(b) Decide which 2-manifolds with boundary can be obtained as W in such a construction.

B2. Let X and Y be manifolds with boundary, and let $f : \partial X \to \partial Y$ be a diffeomorphism. Prove that for each point $x \in \partial X$ there exists a neighborhood $V(x)$ in X and a neighborhood $W(f(x))$ of $f(x)$ in Y, and an extension of f to a diffeomorphsm $g : V(x) \cong W(f(x))$.

B3. Let $f : X^k \to Y^\ell$ be a smooth map, with $W^m \subset Y^\ell$.

Under what additional conditions is $I_2(f, W)$ defined? State the definition carefully, including stating whatever hypotheses and whatever theorems you need for the definition to make sense.

B4. Let S be the standard unit circle in R^2 and let $M = S \times S$. The manifold M lives naturally in R^4, as

$$M = \left\{ (x, y, z, w) \ \mid \ x^2 + y^2 = 1 \text{ and } z^2 + w^2 = 1 \right\} .$$

Define $f : M \to R$ by $f(x, y, z, w) = x + y + z + w$.

DETERMINE all the the points of M at which f is a submersion and the points of M at which f is not a submersion.

B5. (a) Let X be a compact 1-manifold in R^3, and let

$$G = \{u \in R^3 \mid \|u\| = 1 \text{ and } u \in T_x X \text{ for some } x \in X\}.$$

Prove that G is not all of S^2.

(b) Use the result of part (a) to show that there is a plane P in R^3 such that orthogonal projection of X to P is an immersion.

B6. (a) (3 points) Define Morse functions on manifolds.

(b) (5 points) Prove that each smooth map $f : R^n \to R$ can be approximated by a Morse function.

(c) (2 points) Let M be a compact connected surface (2-manifold) without boundary. Let $f : M \to R$ be a Morse function with 3 maxima, 3 minima, and s saddle points. Prove that $s \geq 4$.