

M.S. Comprehensive Exam – Analysis and Topology

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Instructions: Do 3 problems in Part A (Analysis I), 3 problems in Part B (Analysis II), and 4 problems in Part C (topology). This is a “closed book” examination. You should have NO books or papers of your own.

Please indicate here which problems you want to have graded.

Problem: A1 A2 A3 A4 A5 A6
 B1 B2 B3 B4 B5 B6
 C1 C2 C3 C4 C5 C6

Part A: Analysis I

Do three of the following six problems.

A1. Construct a compact subset of \mathbb{R} whose limit points form a countable (infinite) set. Provide arguments.

A2. Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover. Provide arguments.

A3. Let $a_n \geq 0$. Prove that the convergence of the series $\sum_{n=1}^{\infty} a_n$ implies the convergence of

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}.$$

A4. Let f be a real function with domain \mathbb{R} , which has the intermediate value property. Suppose also, for every rational r that the set $\{x \mid f(x) = r\}$ is closed. Prove that f is continuous.

A5. Let f be the function with domain \mathbb{R} defined by the formula

$$f(x) = \begin{cases} \cos \frac{1}{x} & x \neq 0 \\ \frac{1}{2} & x = 0 \end{cases}$$

(A) Does f have the intermediate value property?

(B) Is f the derivative of another function? (Hint: Look carefully at the derivative of the function $x^2 \sin \frac{1}{x}$.)

A6. Give an example of a sequence of real numbers $(a_n)_n$ such that the sequence

$$s_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

converges to 1, but $(a_n)_n$ is divergent.

Part B: Analysis II

Do three of the following six problems.

B1. (1) Give a family \mathbb{P} of partitions of $[0, 1]$ such that

$$\text{Card } \mathbb{P} = \text{Card } \mathbb{N}.$$

(2) Let $f: [0, 1] \rightarrow \mathbb{R}$ be a continuous function. Show that $\int_0^1 |f(x)| dx = 0$ if and only if $f(x) = 0$ for all $x \in [0, 1]$.

B2. (1) Let $f: [0, 1] \rightarrow \mathbb{R}$ be a continuous function. show that f is Riemann integrable on $[0, 1]$ and also show that f has antiderivatives.

(2) Give a function $f: [0, 1] \rightarrow \mathbb{R}$ which is Riemann integrable on $[0, 1]$ but f fails to have antiderivative. Compute $\int_0^1 f(x) dx$ and justify your example.

B3. (1) Give a sequence of functions $f_n, f: [0, 1] \rightarrow \mathbb{R}, n \in \mathbb{N}$ such that $\{f_n\}$ converges to f pointwise but $\{f_n\}$ does not converge uniformly to f on $[0, 1]$. Justify your example.

(2) Let $f_n, f: [0, 1] \rightarrow \mathbb{R}, n \in \mathbb{N}$. If $\{f_n\}$ converges uniformly to f on $[0, 1]$ and $f_n, n \in \mathbb{N}$ are Riemann integrable on $[0, 1]$. Show that f is Riemann integrable on $[0, 1]$.

B4. (1) Give the definition of equicontinuity of a family of real-valued functions on a metric space and give an example of sequence $f_n, g_n: [0, 1] \rightarrow \mathbb{R}, n \in \mathbb{N}$ such that $\{f_n\}$ is equicontinuous on $[0, 1]$ and $\{g_n\}$ is not equicontinuous on $[0, 1]$. Justify your example.

(2) Let K be a compact metric space and let $C(K)$ be the space of all real-valued continuous functions on K with sup norm. Show that every compact set in $C(K)$ is equicontinuous on K .

B5. (1) State the Stone-Weierstrauss theorem.

(2) Let K be a compact metric space and let $C(K)$ be the space of all real-valued continuous functions on K with sup. norm. If X is a linear subspace in $C(K)$ containing the constant functions and X separates the points of K . Show that for any $x_1 \neq x_2$ in K and for any c_1, c_2 , in \mathbb{R} . there exists $f \in X$ such that $f(x_i) = c_i, i = 1, 2$.

B6. (1) Give a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that all directional derivatives of f at 0 exist but f is not differentiable at 0. Justify your example.

(2) State and prove the Banach contraction principle.

Part C: Topology

Do four of the following six problems.

- C1.* Prove the Gluing Lemma: Let A, B be closed subsets of X such that $X = A \cup B$. Let $f : X \rightarrow Y$ be a map such that $f|_A : A \rightarrow Y$ and $f|_B : B \rightarrow Y$ are both continuous. Show that f is continuous.
- C2.* (1) Carefully define product topology on $\prod X_\alpha$.
(2) Let $A \subset X$, and $B \subset Y$. Show that $\overline{A \times B} = \overline{A} \times \overline{B}$.
- C3.* Prove or disprove
- (1) A metric space is a normal space.
 - (2) A metric space has a countable basis.
 - (3) A metric space is covered by countably many compact subsets.
- C4.* Let X be obtained by removing countably many lines from the Euclidean space \mathbb{R}^3 . Show that X is connected.
- C5.* Suppose A is a subspace of X . A continuous function $r : X \rightarrow A$ is a retraction if $r|_A$ is the identity on A . If a retraction $r : X \rightarrow A$ exists, then A is a retract of X . Prove the following, where $A \subset X$:
- (1) A is a retract of X if and only if every continuous function $f : A \rightarrow Y$ can be extended to a continuous function $F : X \rightarrow Y$.
 - (2) Show that if X is Hausdorff and A is a retract of X , then A is closed in X .
- C6.* Let $\mathbb{R}^{\mathbb{N}} = \{f : \mathbb{N} \rightarrow \mathbb{R}\}$ = the set of all functions from the set of natural numbers \mathbb{N} to the real line \mathbb{R} . If $\mathbb{R}^{\mathbb{N}}$ has the product topology, is $\mathbb{R}^{\mathbb{N}}$ connected? What if $\mathbb{R}^{\mathbb{N}}$ has the uniform topology? Prove your claims.