22m015, Fall 2003
Lecture 16 (10/21/03)

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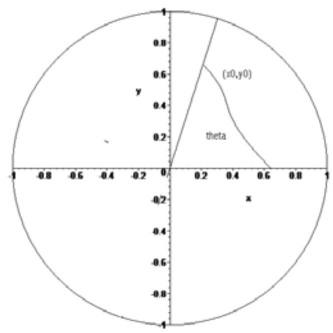
We have already defined what we mean by angle, by the radian measure of an angle and the terminal point of an angle in the unit circle.

Suppose that θ is an angle with terminal point (x_0, y_0).



Definition. $\cos(\theta) = x_0, \sin(\theta) = y_0$.

Immediate Identities:

1. Since the point (x_0, y_0) is in the unit circle, we must have that $x_0^2 + y_0^2 = 1$, thus, if θ is any angle:

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

2. Since the angle $-\theta$ has as terminal point $(x_0, -y_0)$,

$$\cos(-\theta) = \cos(\theta)$$
$$\sin(-\theta) = -\sin(\theta).$$

3. Since adding or subtracting any number of full revolutions about the origin yield the same terminal point, we have, for any $\theta \in \mathbf{R}$:

$$\cos(\theta + 2k\pi) = \cos(\theta), k \in \mathbf{Z}, \theta \text{ in radians.}$$
$$\sin(\theta + 2k\pi) = \sin(\theta), k \in \mathbf{Z}, \theta \text{ in radians.}$$

4. Define the functions $\cos : \mathbf{R} \to \mathbf{R}, \sin : \mathbf{R} \to \mathbf{R}$ in the obvious way (that is $\cos(\theta) = x_0$, from the terminal point and the same for $\sin$), then both of these functions are periodic of period 2π.

5. The sign of $\cos(\theta), \sin(\theta)$ depends on the quadrant where the terminal point lies. If the terminal point lies in the first quadrant, then $\cos(\theta) > 0, \sin(\theta) > 0$, if it lies in the second quadrant, $\cos(\theta) < 0, \sin(\theta) > 0$, if it lies in the third quadrant, $\cos(\theta) < 0, \sin(\theta) < 0$ and if it lies in the fourth quadrant, $\cos(\theta) > 0, \sin(\theta) < 0$.

6.

$$\cos(k\pi) = 1, \text{if } k \in \mathbf{Z} \text{ is even}$$
$$\cos(k\pi) = -1 \text{if } k \in \mathbf{Z} \text{ is odd}$$
$$\sin(k\pi) = 0$$

7. $\cos((\frac{\pi}{2} + k\pi) = 0, k \in \mathbf{Z},$

$$\sin(\frac{\pi}{2} + 2k\pi) = 1, k \in \mathbf{Z}$$
$$\sin(\frac{3}{2}\pi + 2k\pi) = -1, k \in \mathbf{Z}$$

Special angles.

We have seen already that one degree measured in radians has measure $\frac{\pi}{180}$. By the same token, one radian has degree measure equal to $\frac{180}{\pi}$.

We will always use radian measure whenever dealing with trigonometric functions. However, we will make use of the degree measure of an angle to calculate special values of the trigonometric functions, so that we can use facts learned before about triangles and their properties (for example that the angles in a triangle add up to 180°.

1. $\frac{\pi}{4}$, this radian measure corresponds to an angle of 45° and when considering $\sin(\frac{\pi}{4}), \cos(\frac{\pi}{4})$ we first observe that the terminal point lies in the first quadrant, so both are positive, and that, considering the triangle with vertices $(0,0), (\cos(\frac{\pi}{4}), \sin(\frac{\pi}{4})$ and $(\cos(\frac{\pi}{4}), 0)$, we have an isosceles right triangle with hypothenuse equal to 1. Thus $\cos(\frac{\pi}{4}) = \sin(\frac{\pi}{4}) > 0$, and :

$$\cos^2(\frac{\pi}{4}) + \sin^2(\frac{\pi}{4}) = 1$$
$$2\cos^2(\frac{\pi}{4}) = 1$$
$$\cos^2(\frac{\pi}{4}) = \frac{1}{2}, \cos(\frac{\pi}{4}) > 0$$
$$\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}.$$

Thus, $\cos(\frac{\pi}{4}) = \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$. This is the way we will state solutions to our problems, so we will not require "rationalizing the denominator ".

2. $\frac{\pi}{3}$ (the degree measure is 60°). In this case we extend our triangle (by adding a side at the point $(\cos(\frac{\pi}{3}), \sin(\frac{\pi}{3}))$ to an equilateral triangle with the base in the x – axis (draw a picture). Then the height of the triangle is $\sin(\frac{\pi}{3})$ and the midpoint of the base is $\cos(\frac{\pi}{3})$. It follows that $\cos(\frac{\pi}{3}) = \frac{1}{2}$ and so:

$$\cos^2(\tfrac{\pi}{3}) + \sin^2(\tfrac{\pi}{3}) = 1$$

$$\tfrac{1}{4} + \sin^2(\tfrac{\pi}{3}) = 1, \sin(\tfrac{\pi}{3}) > 0$$

$$\sin^2(\tfrac{\pi}{3}) = \tfrac{3}{4}$$

$$\sin(\tfrac{\pi}{3}) = \frac{\sqrt{3}}{2}.$$

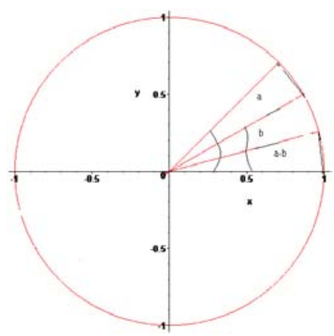
3. $\frac{\pi}{6}$ (this corresponds to a measure of 30°). In this case take the right triangle with vertices at $(0,0), (\cos(\frac{\pi}{6}),0)$ and $(\cos(\frac{\pi}{6}),\sin(\frac{\pi}{6}))$ and "flipped it about the x –axis. You get an equilateral triangle . Since the point $(\cos(\frac{\pi}{6}),0)$ is the midpoint of the side opposite the origin, we must have that $\sin(\frac{\pi}{6}) = \frac{1}{2}$, and by the same type of reasoning we had in 2, $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$.

Theorem. Let a,b be angles. Then

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b).$$

Proof.

We consider first the case when $a \geq b$.



Consider the triangle with vertices at $(0,0), (\cos(b),\sin(b))$ and $(\cos(a),\sin(a))$ and the one with vertices at $(0,0),(1,0)$ and $(\cos(a-b),\sin(a-b))$. The last triangle is a rotation of the first triangle so that the side joining $(0,0)$ and $(\cos(a),\sin(a))$ lies on the x-axis.

It follows that the sides opposite to the origin have equal length, concluding that:

$$\sqrt{(\cos(a)-\cos(b))^2 + (\sin(a)-\sin(b))^2} = \sqrt{(\cos(a-b)-1)^2 + \sin(a-b)^2}$$

$$(\cos(a)-\cos(b))^2 + (\sin(a)-\sin(b))^2 = (\cos(a-b)-1)^2 + \sin(a-b)^2$$

$$\cos^2(a) - 2\cos(a)\cos(b) + \cos^2(b) + \sin^2(a) - 2\sin(a)\sin(b) + \sin^2(b) = \cos^2(a-b) - 2\cos(a-b) + 1 + \text{si}$$

$$2 - 2(\cos(a)\cos(b) + \sin(a)\sin(b) = 2 - 2\cos(a-b)$$

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b).$$

This proves the case $a \geq b$. If $b \geq a$, then:

$$\cos(a-b) = \cos(-(a-b))$$
$$\cos(a-b) = \cos(b-a)$$
$$\cos(a-b) = \cos(b)\cos(a) + \sin(b)\sin(a)$$

and we get the same result.

Consequences:

1. Let a, b be angles. Then

$$\begin{aligned}\cos(a+b) &= \cos(a-(-b)) \\ &= \cos(a)\cos(-b) + \sin(a)\sin(-b) \\ &= \cos(a)\cos(b) - \sin(a)\sin(b).\end{aligned}$$

2. Let θ be an angle. Then;

$$\cos(\theta - \frac{\pi}{2}) = \cos(\theta)\cos(\frac{\pi}{2}) + \sin(\theta)\sin(\frac{\pi}{2}) = \sin(\theta).$$
$$\cos(\theta + \frac{\pi}{2}) = \cos(\theta)\cos(\frac{\pi}{2}) - \sin(\theta)\sin(\frac{\pi}{2}) = -\sin(\theta)$$
$$\sin(\theta + \frac{\pi}{2}) = \cos(\theta + \frac{\pi}{2} - \frac{\pi}{2}) = \cos(\theta)$$
$$\sin(\theta - \frac{\pi}{2}) = \cos(\theta - \frac{\pi}{2} - \frac{\pi}{2}) = \cos(\theta - \pi) = \cos(\theta)\cos(\pi) + \sin(\theta)\sin(\pi)$$
$$= -\cos(\theta).$$

3. Let a, b be angles. Then

$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b), \sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b).$

Proof.

$$\begin{aligned}\sin(a-b) &= \cos(a-b-\frac{\pi}{2}) \\ &= \cos(a-\frac{\pi}{2})\cos(b) + \sin(a-\frac{\pi}{2})\sin(b) \\ &= \sin(a)\cos(b) - \cos(a)\sin(b).\end{aligned}$$

$$\begin{aligned}\sin(a+b) &= \sin(a-(-b)) \\ &= \sin(a)\cos(-b) - \cos(a)\sin(-b) \\ &= \sin(a)\cos(b) + \cos(a)\sin(b).\end{aligned}$$

Examples.

1. Find $\cos(\frac{2}{3}\pi) = \cos(\pi - \frac{\pi}{3}) = \cos(\pi)\cos(\frac{\pi}{3}) + \sin(\pi)\sin(\frac{\pi}{3}) = -\frac{1}{2}.$

2. Let θ be an angle with terminal side in the fourth quadrant and such that $\cos(\theta) = \frac{3}{5}$. Find $\sin(\theta)$.

Since the terminal point is in the fourth quadrant, $\sin(\theta) < 0$.

$$
\begin{aligned}
\cos^2(\theta) + \sin^2(\theta) &= 1, \sin(\theta) < 0 \\
\frac{9}{25} + \sin^2(\theta) &= 1 \\
\sin^2(\theta) &= 1 - \frac{9}{25} = \frac{16}{25} \\
\sin(\theta) &= -\sqrt{\frac{16}{25}} = -\frac{4}{5}.
\end{aligned}
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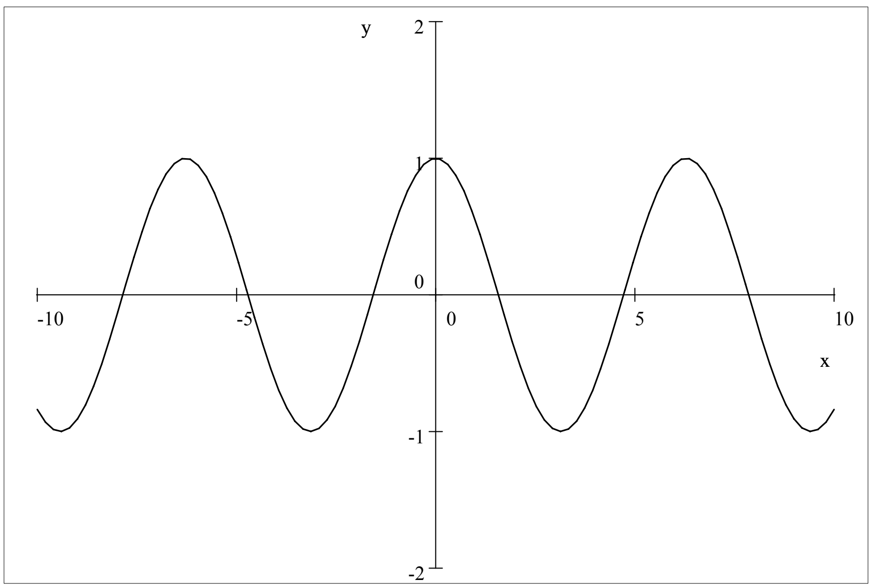
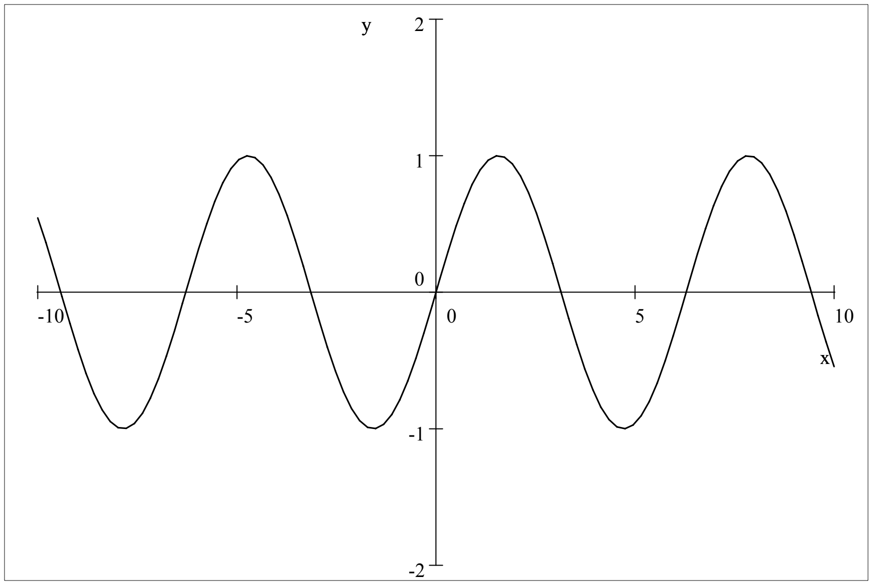
With these tools we can get enough points to graph both sine and cosine.



graph of cosine

For the graph of sine we have:



graph of sine