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BOUNDARY INTEGRAL EQUATION METHODS FOR SOLVING LAPLACE'S EQUATION WITH NONLINEAR BOUNDARY CONDITIONS: THE SMOOTH BOUNDARY CASE

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ABSTRACT. A nonlinear boundary value problem for Laplace's equation is solved numerically by using a reformulation as a nonlinear boundary integral equation. Two numerical methods are proposed and analyzed for discretizing the integral equation, both using product integration to approximate the singular integrals in the equation. The first method uses the product Simpson's rule, and the second is based on trigonometric interpolation. Iterative methods (including two-grid methods) for solving the resulting nonlinear systems are also discussed extensively. Numerical examples are included.

1. INTRODUCTION

Consider solving the nonlinear boundary value problem

$$\Delta u(P) = 0, \qquad P \in D, \tag{1.1}$$

$$\frac{\partial u(P)}{\partial n_P} = -g(P, u(P)) + f(P), \qquad P \in \Gamma = \partial D. \tag{1.2}$$

We study the numerical solution of a nonlinear boundary integral equation reformulation of this problem, a reformulation that has been studied recently in Ruotsalainen and Wendland [8]. In (1.1), we assume D is a bounded, simply connected open region in $\mathbb{R}^2$ with a smooth boundary Γ, and we seek a solution $u \in C^2(D) \cap C^1(\overline{D})$. Our numerical methods generalize to other problems, for example exterior problems, but these are not considered here. Also in (1.2), n_P denotes the exterior unit normal to Γ at P, and the function f is assumed given and continuous on Γ. The function $g(P, v)$ is assumed to be continuous for $(P, v) \in \Gamma \times \mathbb{R}$, although this can be relaxed. Further assumptions on g are given later.

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Using Green's representation formula for harmonic functions, the function u satisfies

$$\tag{1.3} \begin{aligned} u(P) = {} & \frac{1}{2\pi}\int_\Gamma u(Q)\frac{\partial}{\partial n_Q}[\log|P-Q|]\,d\sigma(Q) \\ & - \frac{1}{2\pi}\int_\Gamma \frac{\partial u(Q)}{\partial n_Q}\log|P-Q|\,d\sigma(Q) \end{aligned}$$

for all $P \in D$. Letting P tend to a point on Γ, and using the boundary condition in (1.2), we obtain

$$\tag{1.4} \begin{aligned} & u(P) - \frac{1}{\pi}\int_\Gamma u(Q)\frac{\partial}{\partial n_Q}[\log|P-Q|]\,d\sigma(Q) \\ & \quad - \frac{1}{\pi}\int_\Gamma g(Q, u(Q))\log|P-Q|\,d\sigma(Q) \\ & = -\frac{1}{\pi}\int_\Gamma f(Q)\log|P-Q|\,d\sigma(Q), \qquad P \in \Gamma. \end{aligned}$$

This can be solved for u on Γ, the normal derivative can then be obtained from (1.2), and finally the representation (1.3) gives $u(P)$ for $P \in D$. One very important requirement for the use of (1.3)–(1.4) is that the *transfinite diameter* of Γ, denoted by C_Γ, cannot be equal to 1. If $C_\Gamma = 1$ in a problem, then (1.1)–(1.2) can be redefined on a rescaled region D in such a way that the new $C_\Gamma \neq 1$. For a more extensive discussion of transfinite diameter, see Yan and Sloan [9].

Our development follows that of Ruotsalainen and Wendland [8], who give results on both the theoretical solvability of (1.4) and on its numerical analysis. We do not examine the solvability of (1.4), but simply assume it is uniquely solvable in a suitable sense; the precise assumptions are given later in §2. Ruotsalainen and Wendland [8, Theorem 1] show unique solvability of (1.4) under the additional assumption that

$$\tag{1.5} 0 < l \leq \frac{\partial}{\partial v}g(P, v) \leq L < \infty, \qquad P \in \Gamma,\ v \in \mathbb{R},$$

for some constants l, L. Other existence proofs can be based on contractive mapping arguments, under other suitable assumptions on g and f.

In §2, we define a numerical approximation of (1.4) by using the trapezoidal rule to approximate the double layer integral in (1.4), and product integration with piecewise polynomial interpolation to approximate the single layer integral. An error analysis is given using the general framework of Atkinson [2], and the ideas are illustrated numerically with a product Simpson's rule. In §3, we consider some two-grid iteration methods for solving the nonlinear system of equations that arises in the discretizations of §2. These generalize methods introduced in Atkinson [3].

To take maximum advantage of the smoothness of the boundary Γ, we introduce another approximation in §4. It is based on approximating the single layer integral operator in (1.4) by product integration with trigonometric interpolation. The error analysis is slightly more difficult than that for the piecewise polynomial product integration of §2, but the convergence is much faster, as is illustrated in the numerical examples.

2. Piecewise polynomial product integration

We discretize the nonlinear integral equation (1.4) by approximating the integrals in it and then reduce the equation to a finite system of nonlinear equations. This system must be solved by iteration, and we discuss some two-grid iteration methods in §3. We begin by introducing some operator notation.

Let $\mathscr{K}$ denote the double layer integral operator

$$(2.1)\qquad (\mathscr{K}v)(P) = \frac{1}{\pi}\int_\Gamma v(Q)\frac{\partial}{\partial n_Q}[\log|P-Q|]\,d\sigma(Q), \qquad P\in\Gamma,$$

and let $\mathscr{S}$ denote the single layer integral operator

$$(2.2)\qquad (\mathscr{S}v)(P) = -\frac{1}{\pi}\int_\Gamma v(Q)\log|P-Q|\,d\sigma(Q), \qquad P\in\Gamma.$$

Also introduce the nonlinear operator

$$(2.3)\qquad (\mathscr{G}v)(P) = g(P, v(P)), \qquad P\in\Gamma.$$

We assume this is well defined from $C(\Gamma)$ into $C(\Gamma)$, with additional assumptions given later. The assumption that $\mathscr{G}$ is defined on all of $C(\Gamma)$ can be relaxed without any essential difficulty, but we do not consider this here. See Krasnoselskii [6, pp. 20–32] for a more formal discussion of such operators $\mathscr{G}$.

The integral equation (1.4) can now be written symbolically as

$$(2.4)\qquad u - \mathscr{K}u + \mathscr{S}\mathscr{G}(u) = \mathscr{S}f.$$

To approximate this equation, we replace $\mathscr{K}$ and $\mathscr{S}$ with a sequence of numerical integral operators $\mathscr{K}_n$ and $\mathscr{S}_n$. Introduce the parametrization

$$r(t) = (\xi(t), \eta(t)), \qquad 0\le t\le 2\pi,$$

of the boundary Γ. Assume $r\in C_p^\infty(2\pi)$, the space of C^∞ 2π-periodic functions, and also assume

$$|r'(t)|\neq 0, \qquad 0\le t\le 2\pi.$$

The assumption of C^∞ continuity is just for simplicity, and one can prove convergence of our numerical methods under the weaker assumption $r\in C_p^2(2\pi)$, although with a much slower rate of convergence.

The double layer operator $\mathscr{K}$ can now be written

$$(2.5)\qquad (\mathscr{K}v)(t) = \frac{1}{\pi}\int_0^{2\pi}\frac{\eta'(s)[\xi(s)-\xi(t)] - \xi'(s)[\eta(s)-\eta(t)]}{[\xi(s)-\xi(t)]^2 + [\eta(s)-\eta(t)]^2}v(s)\,ds$$

for $v\in C_p(2\pi)$. When $s = t$, the kernel function has the value

$$\frac{\xi'(t)\eta''(t) - \eta'(t)\xi''(t)}{2[\xi'(t)^2 + \eta'(t)^2]}.$$

To approximate this integral, use the trapezoidal rule, midpoint rule, Simpson's rule, or some other composite integration rule. Let $\mathscr{K}_n v$ denote the numerical integral approximation of (2.5) with n subdivisions of $[0, 2\pi]$. The error in $\mathscr{K}_n v$ satisfies

$$\|\mathscr{K} v - \mathscr{K}_n v\|_\infty \le \frac{c_q}{n^q}\|v\|_{H^q(2\pi)}, \qquad v \in H^q(2\pi),\ q > \frac{1}{2}. \tag{2.6}$$

The notation $H^q(2\pi)$ denotes the Sobolev space of index $q \ge 0$ with functions that are 2π-periodic.

The single layer operator $\mathscr{S}$ is written as

$$(\mathscr{S} v)(t) = -\frac{1}{\pi}\int_0^{2\pi} v(s)|r'(s)|\log|r(t) - r(s)|\,ds. \tag{2.7}$$

We use product integration to approximate $\mathscr{S} v$, but it will be convenient to first modify (2.7). Write it as

$$\begin{aligned}(\mathscr{S} v)(t) = &-\frac{1}{\pi}\int_0^{2\pi} v(s)|r'(s)| \\ &\qquad \times \{\log|t-s| + \log|2\pi - s + t| + \log|2\pi - t + s|\}\,ds \\ &-\frac{1}{\pi}\int_0^{2\pi} v(s)|r'(s)|\log\left[\frac{|r(t) - r(s)|}{|t-s|(2\pi - s + t)(2\pi - t + s)}\right] ds.\end{aligned} \tag{2.8}$$

For $n \ge 0$, define $h = 2\pi/n$ and $t_{j,n} = t_j = jh$. To make clearer the definition of our numerical method, we define the *product quadrature rule* for approximating $\mathscr{S}$. Let n be even. The last integral on the right side of (2.8) has a kernel function that is C^∞ on $[0, 2\pi]$. We approximate this integral by the regular Simpson's rule; and the error is $O(h^4)$ if $v \in C_p^4(2\pi)$.

For the first integral on the right side of (2.8), replace $v(s)|r'(s)|$ by its piecewise quadratic interpolant on the mesh $\{t_0, t_1, \dots, t_n\}$. The resulting integration can be done explicitly, and a careful consideration of the formulas will lead to quite inexpensive implementations. The use of product integration in solving integral equations is discussed extensively in Atkinson [4, pp. 106–123]. Denote the combined approximation of the right-hand integrals of (2.8) by $\mathscr{S}_n v(t)$. For the error, the results in de Hoog and Weiss [5] can be used to prove

$$\|\mathscr{S} v(t) - \mathscr{S}_n v(t)\|_\infty \le c\frac{\log n}{n^4}, \qquad v \in C^4[0, 2\pi]. \tag{2.9}$$

We approximate the nonlinear equation (2.4) by

$$u_n - \mathscr{K}_n u_n + \mathscr{S}_n \mathscr{G}(u_n) = \mathscr{S}_n f, \tag{2.10}$$

and we seek a solution $u_n \in C_p(2\pi)$. This is a Nyström method for solving the nonlinear equation (2.4), and a complete framework and error analysis for such methods is given in Atkinson [2]. Equation (2.10) is equivalent to a finite nonlinear system, which is given below in (2.13).

Let the linear operators $\mathscr{K}_n$ and $\mathscr{S}_n$ be written as

$$(\mathscr{K}_n v)(t) = \sum_{j=0}^{n} w_j K(t, t_j) v(t_j), \qquad t \in [0, 2\pi], \tag{2.11}$$

$$(\mathscr{S}_n v)(t) = \sum_{j=0}^{n} \omega_j(t) v(t_j), \qquad t \in [0, 2\pi]. \tag{2.12}$$

The kernel function $K(t, s)$ is given in (2.5), and the weights $\{\omega_j(t)\}$ are obtained from the approximation of the right-hand integrals in (2.8). For (2.11), we will use Simpson's rule as the quadrature method, partially to be consistent with the earlier definition given for (2.12). Just as with the linear Nyström method (see Atkinson [4, p. 88]), equation (2.10) is equivalent to a finite system of equations,

$$\begin{aligned} u_n(t_i) - \sum_{j=0}^{n} w_j K(t_i, t_j) u_n(t_j) + \sum_{j=0}^{n} \omega_j(t_i) g(t_j, u_n(t_j)) \\ = \sum_{j=0}^{n} \omega_j(t_i) f(t_j). \end{aligned} \tag{2.13}$$

The grid function that solves (2.13) is extended to a function on $[0, 2\pi]$ by means of the Nyström interpolation formula

$$\begin{aligned} u_n(t) = \sum_{j=0}^{n} w_j K(t, t_j) u_n(t_j) \\ + \sum_{j=0}^{n} \omega_j(t)[-g(t_j, u_n(t_j)) + f(t_j)]. \end{aligned} \tag{2.14}$$

We use this formula in the two-grid iteration method presented in §3.

The error analysis of (2.10) can be carried out within the framework of [2]. Write (2.4) and (2.10) in the shortened form

$$u = \mathscr{L}(u), \tag{2.15}$$

$$u_n = \mathscr{L}_n(u_n), \tag{2.16}$$

respectively. With the assumption of $\mathscr{G}$ following (2.3), the operator $\mathscr{L}$ is compact from $C(\Gamma)$ into $C(\Gamma)$. We must also have that $\mathscr{L}$ is continuous, and thus we assume

[A1] $\mathscr{G}: C(\Gamma) \to C(\Gamma)$ is continuous.

With this and the known properties of $\mathscr{K}$ and $\mathscr{S}$, it follows that $\mathscr{L}$ is completely continuous (compact and continuous) from $C(\Gamma)$ into $C(\Gamma)$. The assumption [A1] is true if $g(P, v)$ satisfies the Lipschitz condition

$$|g(P, v_1) - g(P, v_2)| \le c|v_1 - v_2|^\lambda, \qquad P \in \Gamma,$$

for some exponent $\lambda \in (0, 1]$ and constant $c > 0$.

The framework of [2] assumes that $\{\mathscr{L}_n\}$ satisfies the following four properties:

[H1] $\mathscr{L}$ and $\mathscr{L}_n$, $n \geq 1$, are completely continuous operators on a Banach space $\mathscr{X}$ into itself.

[H2] $\{\mathscr{L}_n\}$ is a collectively compact family, i.e., for every bounded set B in $\mathscr{X}$, the set $\bigcup_1^\infty \mathscr{L}_n(B)$ has compact closure in $\mathscr{X}$.

[H3] For every $v \in \mathscr{X}$,

$$\mathscr{L}_n v \to \mathscr{L} v \quad \text{as } n \to \infty.$$

[H4] At each $v \in \mathscr{X}$, $\{\mathscr{L}_n\}$ is an equicontinuous family.

These assumptions are true for our approximations $\mathscr{L}_n$. In [H1], we let $\mathscr{X} = C(\Gamma)$. The complete continuity of $\mathscr{L}_n$ follows from [A1] and the compactness of the finite-rank operators $\mathscr{K}_n$ and $\mathscr{S}_n$. For [H2], the proof of the collective compactness of $\{\mathscr{L}_n\}$ follows from that of $\{\mathscr{K}_n\}$ and $\{\mathscr{S}_n\}$, and the latter are well-known results (e.g., see Atkinson [4, pp. 97, 108]). The proof of [H3] again follows from the same result for $\{\mathscr{K}_n\}$ and $\{\mathscr{S}_n\}$, along with the fact that $\mathscr{G}(v) \in C(\Gamma)$. For [H4], use

$$\|\mathscr{L}_n(v_1) - \mathscr{L}_n(v_2)\|_\infty \leq \|\mathscr{K}_n\| \, \|v_1 - v_2\|_\infty + \|\mathscr{S}_n\| \, \|\mathscr{G}(v_1) - \mathscr{G}(v_2)\|_\infty .$$

The families $\{\mathscr{K}_n\}$ and $\{\mathscr{S}_n\}$ are uniformly bounded, and then [A1] completes the proof of equicontinuity. We state the following existence theorem for approximate solutions u_n without proof. It is a direct statement from [2, Theorem 3].

Theorem 1. *Let u_0 be an isolated solution of* (2.4), *with no other solutions in the ball*

$$B(u_0, r_0) = \{v \mid \|u_0 - v\|_\infty \leq r_0\}$$

for some $r_0 > 0$. In addition, assume u_0 has nonzero index as a solution of (2.4). *Then for every $0 < r \leq r_0$, there is an integer $N(r)$ such that for $n \geq N(r)$*

(i) *the approximating equation* (2.10) *has no solution in the annular region*

$$\{v \mid r \leq \|u_0 - v\| \leq r_0\};$$

(ii) *equation* (2.10) *has at least one solution u_n inside $B(u_0, r)$.*

As a consequence, the solutions u_n of (2.10) exist for all sufficiently large n, and they converge to u_0 as $n \to \infty$. (This result allows the possibility that u_0 is a "multiple root" of (2.4), with several distinct and nearly equal approximate solutions u_n for each n, all converging to u_0.)

Remark. Equation (2.15) has an isolated solution u_0 of *index zero* if and only if the equation satisfies the following property:

> There exists some open neighborhood N of u_0 such that for every $\delta > 0$, there exists $\mathscr{L}_\delta$ defined on $\overline{N}$ (the closure of N) with
>
> $$\|\mathscr{L}(u) - \mathscr{L}_\delta(u)\| \leq \delta, \qquad u \in \overline{N},$$
>
> and with the equation $u = \mathscr{L}_\delta(u)$ having no solutions in $\overline{N}$.

Thus, a solution u_0 has nonzero index if and only if the existence of solutions to the equation is stable with respect to small perturbations of the equation. In general, we would consider solving only those equations in which the solution u_0 possesses this type of stability.

To obtain results on the rate of convergence of u_n to u_0, an additional assumption is needed for the operators $\mathscr{L}$ and $\mathscr{L}_n$:

[H5] For a given solution u_0 of $u = \mathscr{L}(u)$ and $r > 0$, assume $\mathscr{L}$ and $\mathscr{L}_n$, $n \geq 1$, are twice Fréchet-differentiable on $B(u_0, r)$. Moreover, assume

$$\|\mathscr{L}''(u)\|, \|\mathscr{L}_n''(u)\| \leq c, \qquad u \in B(u_0, r), \; n \geq 1. \tag{2.17}$$

In our case, this will be satisfied if we assume

[A2] The function $g(P, v)$ is twice differentiable with respect to v, for all $P \in \Gamma$ and all $v \in \mathbb{R}$, and it satisfies

$$\left|\frac{\partial^2 g(P, v)}{\partial^2 v}\right| \leq c(a, b), \qquad P \in \Gamma, \; a \leq v \leq b, \tag{2.18}$$

for every finite interval $[a, b]$, with $c(a, b)$ a constant.

This is easily satisfied with most functions g that one is likely to encounter. From [2, Theorem 4], we have the following.

Theorem 2. *Let u_0 be an isolated fixed point of $\mathscr{L}$, say in $B(u_0, r)$, and assume g satisfies* [A1] *and* [A2]. *In addition, assume* 1 *is not an eigenvalue of $\mathscr{L}'(u_0)$. Then u_0 is an isolated solution of $u = \mathscr{L}(u)$ of nonzero index. Moreover, there are $0 < \varepsilon \leq r$ and $N > 0$ such that for every $n \geq N$, $u = \mathscr{L}_n(u)$ has a unique solution $u_n \in B(u_0, \varepsilon)$. Also, there is a constant $\gamma > 0$ such that*

$$\|u_0 - u_n\|_\infty \leq \gamma \|\mathscr{L}(u_0) - \mathscr{L}_n(u_0)\|_\infty, \qquad n \geq N. \tag{2.19}$$

This bounds the speed of convergence of u_n to u_0.

With the earlier results (2.6) and (2.9) on the errors in the discretizations $\mathscr{K}_n$ and $\mathscr{S}_n$, we can use (2.19) to bound the error in our method (2.10) based on piecewise quadratic product integration and Simpson's rule.

Theorem 3. *Assume the function $g(P, v)$ satisfies the property that*

[A3] *$u \in C_p^4(2\pi)$ implies $g(\cdot, u(\cdot)) \in C_p^4(2\pi)$.*

Assume [A1], [A2], *and the hypotheses of Theorem* 2. *Finally, assume the solution u_0 of* (2.4) *is in $C_p^4(2\pi)$. Then the numerical solutions u_n satisfy*

$$\|u_0 - u_n\|_\infty \leq \frac{c \log n}{n^4} \tag{2.20}$$

for all sufficiently large n.

Remark. The assumption [A3] can be replaced by the much weaker assumption that u_0 and $g(\cdot, u_0(\cdot))$ belong to $C_p^4(2\pi)$.

Examples. We solve (1.1)–(1.2) with the two choices of g used in Ruotsalainen and Wendland [8]. These are

$$g_1(P, v) = v + \sin v, \tag{2.21}$$

$$g_2(P, v) = |v|v^3. \tag{2.22}$$

The function $g_1 \in C^\infty(\mathbb{R})$, and thus [A1]–[A3] are satisfied easily. The function g_2 is only three times continuously differentiable, with the fourth derivative satisfying a Lipschitz condition. For practical purposes, g_2 satisfies [A3]; this is verified empirically. We choose a known true solution,

$$u_0(x, y) = e^x \cos y, \tag{2.23}$$

and the function $f = \partial u_0(P)/\partial n + g(P, u_0(P))$ is calculated accordingly. For the region D, we use the elliptical region

$$(x/a)^2 + (y/b)^2 < 1$$

with various values of (a, b). The values in Tables 1 and 2 use $(a, b) = (1, 2)$.

Numerical results are given in Tables 1 and 2 for solving (1.1)–(1.2) with g_1 and g_2, respectively. The numerical method is (2.10), with the use of Simpson's rule and the quadratic product integration. The error $\|u - u_0\|_\infty$ is the maximum error at the node points on Γ. According to (2.20), we should have the error decreasing by a factor of about 16 when n is doubled. This is verified empirically for larger values of n, as can be seen in the column *Rate*, the quotient of successive errors.

TABLE 1
Using (2.10) *to solve* (1.4) *with* $g = g_1$

n	$\|u_0 - u_n\|_\infty$	*Rate*	*IT*	*COND*
4	9.93E − 1		6	5.8
8	4.44E − 1	2.2	6	7.3
16	9.00E − 2	4.9	6	8.2
32	5.29E − 3	17.0	6	8.5
64	3.04E − 4	17.4	6	9.1

TABLE 2
Using (2.10) *to solve* (1.4) *with* $g = g_2$

n	$\|u_0 - u_n\|_\infty$	*Rate*	*IT*	*COND*
4	5.89E − 1		8	2300
8	2.81E − 1	2.1	8	750
16	5.64E − 2	5.0	9	360
32	2.83E − 3	19.9	9	720
64	1.47E − 4	19.3	9	930

The nonlinear system (2.13) was solved iteratively using Newton's method:

$$u_n^{(k+1)} = u_n^{(k)} - [I - \mathscr{L}_n'(u_n^{(k)})]^{-1}[u_n^{(k)} - \mathscr{L}_n(u_n^{(k)})]. \tag{2.24}$$

The initial guess chosen was

$$u_n^{(0)} = u_0 + 1. \tag{2.25}$$

We iterated until the relative correction satisfied

$$\frac{\|u_n^{(k)} - u_n^{(k-1)}\|_\infty}{\|u_n^{(k)}\|_\infty} \le 10^{-14}. \tag{2.26}$$

All norms are maximum norms over the values at the nodes on Γ. The column *IT* gives the lowest value of k for which (2.26) was satisfied. The column *COND* gives the LINPACK condition number for the matrix associated with $I - \mathscr{L}_n'(u_n^{(k)})$ for the final iterate. Note the values of *IT* are essentially constant with increasing n. This is an illustration of the *mesh independence principle* discussed in Allgower et al. [1]. All of the numerical examples of this paper were computed on an 80286/287 microcomputer.

3. Iterative solution of discretized equation

The approximating integral equation (2.10) of the preceding section can be solved using Newton's method (2.24), as was done in the numerical examples of the last section. This is rapidly convergent, but it is also quite costly. Each iteration involves solving a system of order $n+1$ at a cost of about $2n^3/3$ arithmetic operations. In this section, we consider other iteration methods that are less costly. Another source of inefficiency in constructing Tables 1 and 2 was that the nonlinear system was solved to much greater accuracy than justified by the size of $\|u_0 - u_n\|_\infty$, mostly to illustrate the iteration method for different values of n. A practical program would attempt to iterate only until $\|u_n - u_n^{(k)}\|_\infty$ was comparable to $\|u_0 - u_n\|_\infty$.

The simplest modification of Newton's method (2.24) is to fix the derivative matrix $I - \mathscr{L}_n'(u_n^{(k)})$ for iterates of index $k \ge \hat{k}$, for some $\hat{k} \ge 0$. The iteration then becomes

$$\begin{gathered} r_n^{(k)} = u_n^{(k)} - \mathscr{L}_n(u_n^{(k)}), \\ [I - \mathscr{L}_n'(u_n^{(\hat{k})})]\delta_n^{(k)} = -r_n^{(k)}, \\ u_n^{(k+1)} = u_n^{(k)} + \delta_n^{(k)}. \end{gathered} \tag{3.1}$$

For iterates $u_n^{(k)}$ with $k > \hat{k}$, the cost of (3.1) is $O(n^2)$ operations per iterate, since the LU factorization of $I - \mathscr{L}_n'(u_n^{(\hat{k})})$ will have already been computed.

The rate of convergence of (3.1) will only be linear, in contrast to the quadratic convergence of the Newton method (2.24). The iterates satisfy

$$\begin{aligned} u_n - u_n^{(k+1)} = {} & \{I - [I - \mathscr{L}_n'(u_n^{(\hat{k})})]^{-1}[I - \mathscr{L}_n'(u_n)]\}[u_n - u_n^{(k)}] \\ & + O(\|u_n - u_n^{(k)}\|^2). \end{aligned} \tag{3.2}$$

This gives linear convergence, with the rate improving as $u_n^{(\hat{k})}$ approaches u_n. The number of iterates needed for convergence is greater than for Newton's method; but the number needed for an iteration error of at most a given $\delta > 0$ can be shown to be bounded independent of n. A condition for choosing $\hat{k}$ is to use the smallest k for which

$$\frac{\|u_n^{(k)} - u_n^{(k-1)}\|_\infty}{\|u_n^{(k)}\|} \le \varepsilon, \tag{3.3}$$

where ε is given (say $\varepsilon = .1$). We have used this condition in the following numerical examples.

Examples. We solve the problem (1.1)–(1.2) with the functions g of (2.21) and (2.22), and we use the modified Newton method (3.1). The test (3.3) was used to determine $\hat{k}$, and the initial guess $u_n^{(0)}$ was (2.25), the same as used in Tables 1 and 2 for Newton's method. Again, we iterated until the relative error test (2.26) was satisfied. In Tables 3 and 4, the column *IT* is the total number of iterates computed, and $\hat{k}$ is the index used in (3.2) and discussed preceding (3.3). The column *Ratio* gives the empirically observed convergence rate

$$\frac{\|u_n^{(k)} - u_n^{(k-1)}\|_\infty}{\|u_n^{(k-1)} - u_n^{(k-2)}\|_\infty}. \tag{3.4}$$

In Tables 3 and 4, the parameters for the elliptical boundary Γ are $(a, b) = (1, 2)$.

Two-grid methods. The modified Newton method still requires $O(n^3)$ arithmetic operations, because the LU factorization of the derivative matrix $I - \mathscr{L}_n'(u)$ must be computed at least once (and usually more). To avoid this, we use the LU factorization of a lower-order derivative matrix $I - \mathscr{L}_m'(u_m)$,

TABLE 3

The modified Newton method: Case $g = g_1$

ε	n	IT	$\hat{k}$	*Ratio*
.1	4	12	2	.0571
	8	8	3	.00514
	16	11	3	.0306
	32	10	3	.0180
	64	10	3	.0186
.01	4	7	3	.00101
	8	8	3	.00514
	16	7	4	.0048
	32	7	4	.00017
	64	7	4	.00018

TABLE 4
The modified Newton method: Case $g = g_2$

ε	n	IT	$\hat{k}$	*Ratio*
.1	4	50	3	.547
	8	16	5	.0776
	16	29	4	.311
	32	30	4	.321
	64	30	4	.324
.01	4	13	5	.0260
	8	9	6	.00015
	16	10	6	.00134
	32	10	6	.00157
	64	10	6	.00163

$m \ll n$, to construct an approximation to the solution of the linear system occurring in (2.24) or (3.1). These ideas for solving nonlinear equations were introduced in Atkinson [3], and we extend that discussion.

While ordinarily we are interested only in solving for the grid function on $\{t_{i,n}\}$ that solves the nonlinear system (2.13), we will use the linear analogue of the Nyström interpolation formula (2.14) to move between grid functions defined on $\{t_{i,n}\}$ and $\{t_{i,m}\}$. For this reason, and because the error analysis is easier in $C(\Gamma)$, we use the function space setting of $C(\Gamma)$ with the operators $\mathscr{L}_n$ and $\mathscr{L}'_n$, rather than limiting ourselves to the solution of (2.13) on just the mesh $\{t_{i,n}\}$. Later we give an explicit form of the second of our two-grid methods for solving the nonlinear system (2.13).

Our first two-grid method uses the simple approximation

$$[I - \mathscr{L}'_n(u_n^{(k)})]^{-1} \doteq [I - \mathscr{L}'_m(u_m)]^{-1}. \tag{3.5}$$

The iteration method becomes

$$\begin{aligned} r_n^{(k)} &= u_n^{(k)} - \mathscr{L}_n(u_n^{(k)}), \\ [I - \mathscr{L}'_m(u_m)]\delta_n^{(k)} &= -r_n^{(k)}, \\ u_n^{(k+1)} &= u_n^{(k)} + \delta_n^{(k)}. \end{aligned} \tag{3.6}$$

When this method is applied to solve the system (2.13), the operations count can be shown to be $O(n^2)$ per iteration.

The iterates $u_n^{(k)}$ satisfy the recursion relation

$$\begin{aligned} u_n - u_n^{(k+1)} &= M_{m,n}[u_n - u_n^{(k)}] + O(\|u_n - u_n^{(k)}\|^2), \\ M_{m,n} &= I - [I - \mathscr{L}'_m(u_m)]^{-1}[I - \mathscr{L}'_n(u_n)]. \end{aligned} \tag{3.7}$$

It can be shown that

$$\underset{m\to\infty}{\text{Limit}}\ \underset{n>m}{\text{Sup}}\|M_{m,n}^2\| = 0. \tag{3.8}$$

TABLE 5

The two-grid method (3.6): *Case* $g = g_1$

a	b	m	n	IT	*Ratio*
1.0	2.0	8	16	63	.622
		16	32	29	.381
		16	64	29	.379
.5	1.0	8	16	25	.305
		8	32	25	.299
		8	64	25	.308
		16	32	16	.183
		16	64	16	.182

Combined with (3.7), this will show that $u_n^{(k)} \to u_n$ as $k \to \infty$, provided m is chosen sufficiently large. The proof of (3.8) is fairly straightforward. It uses the assumption [A2], the bound

$$\|\mathscr{L}'_m(u) - \mathscr{L}'_m(v)\| \leq \operatorname*{Max}_{0 \leq \theta \leq 1} \|\mathscr{L}''_m(\theta u + (1-\theta)v)\|,$$

and the proof in Atkinson [4, p. 139].

Example. We solve the first of the problems considered in the earlier examples of this section, where the numerical results were given in Table 3. We allow the boundary ellipse parameters (a, b) to vary, however, The results are given in Table 5. They show a linear rate of convergence, with an improvement as the course mesh parameter m is increased.

We have omitted results for the case $g = g_2$, mostly because they are so poor. Only when the boundary Γ becomes small (a and b become small) does the iteration method become convergent for values of $m \leq 16$. A critical factor in $\|M^2_{m,n}\|$ in (3.8) is $\|[\mathscr{L}'_m(u_m) - \mathscr{L}'(u_m)]\mathscr{L}'_m(u_m)\|$, and it can be shown to be $O(m^{-1.5})$. This is reflected in the relatively small decrease in the value of *Ratio* when m is doubled in Table 5. From earlier results in Tables 1 through 4, it can be seen that the case $g = g_2$ is more badly behaved than is the case $g = g_1$, and this is reflected in the behavior of the iteration method (3.6) when applied to $g = g_2$. With $(a, b) = (.5, .5)$ and $g = g_2$, the iteration (3.6) converges with $m = 16$ and $n > m$. With $(m, n) = (16, 32)$ and $(16, 64)$, we have $Ratio \doteq 0.74$.

For a second two-grid method, we use the approximation

$$[I - \mathscr{L}'_n(u_n^{(k)})]^{-1} \doteq I + [I - \mathscr{L}'_m(u_m)]^{-1}\mathscr{L}'_n(u_n^{(k)}). \tag{3.9}$$

The iteration method becomes

$$\begin{aligned} r_n^{(k)} &= u_n^{(k)} - \mathscr{L}_n(u_n^{(k)}), \\ [I - \mathscr{L}'_m(u_m)]\delta_n^{(k)} &= \mathscr{L}'_n(u_n^{(k)})r_n^{(k)}, \\ u_n^{(k+1)} &= u_n^{(k)} - r_n^{(k)} - \delta_n^{(k)}. \end{aligned} \tag{3.10}$$

The approximation in (3.9) comes from the theory of collectively compact operator approximations for linear operators; for example, see Atkinson [4, p. 94]. For the development of the iteration method (3.10) for linear operators, see Atkinson [4, p. 142] or Atkinson [3].

The iterates $u_n^{(k)}$ satisfy the recursion relation

$$\begin{aligned} u_n - u_n^{(k+1)} &= M_{m,n}[u_n - u_n^{(k)}] + O(\|u_n - u_n^{(k)}\|^2), \\ M_{m,n} &= [I - \mathscr{L}_m'(u_m)]^{-1}[\mathscr{L}_n'(u_n) - \mathscr{L}_m'(u_m)]\mathscr{L}_n'(u_n). \end{aligned} \tag{3.11}$$

It can be shown that

$$\operatorname*{Limit\,Sup}_{m\to\infty\ n>m}\|M_{m,n}\| = 0. \tag{3.12}$$

The proof is fairly straightforward, just as for (3.8), and we omit it. From (3.12), it follows easily that

$$\|u_n - u_n^{(k+1)}\| \le (\|M_{m,n}\| + \varepsilon_n)\|u_n - u_n^{(k)}\| \tag{3.13}$$

with $\varepsilon_n \to 0$ as $n \to \infty$. It can be shown that

$$\|M_{m,n}\| = O(m^{-1.5}). \tag{3.14}$$

The proof is much the same as for (4.12) in §4, and thus we omit it here.

Examples. We repeat the cases used in the preceding example, but now using the two-grid method (3.10). The results are given in Table 6, and again they show a linear rate of convergence. [The value of *Ratio* for the case $(a, b) = (1, 2)$ and $m = 8$ oscillated between two values, and the geometric mean of these values is given in the table.] Considering the results of Tables 5 and 6, the two-grid method (3.10) is superior to the two-grid method (3.6) when the rates of linear convergence are compared. This has been observed in general, and some theoretical support can be given for it.

We also solved the integral equation (2.4) with $g = g_2$; and as was true with (3.6), the results were much poorer than those in $g = g_1$. There was no

TABLE 6
The two-grid method (3.10): *Case* $g = g_1$

a	b	m	n	IT	*Ratio*
1.0	2.0	8	16	47	.50*
		16	32	17	.186
		16	64	17	.178
0.5	1.0	8	16	15	.132
		8	32	15	.124
		8	64	15	.128
		16	32	9	.045
		16	64	9	.042

convergence for $(a, b) = (1, 2)$ or $(.5, 1)$, with $m \leq 16$. For $(a, b) = (.5, .5)$ and $g = g_2$, the iteration (3.10) converges for $m = 16$ and $n > m$. With $(n, m) = (16, 32)$ and $(16, 64)$, we have *Ratio* $= .795$ and $.754$, respectively. This is slightly worse than the analogous results for method (3.6), but it is expected that the method (3.10) will be superior to (3.6) for larger values of m.

Finite Algorithm. We reduce the two-grid iteration (3.10) for solving the function space equation (2.10) to an algorithm for the finite system of nonlinear equations (2.13). Calculate:

1. $r_n(t_{in}) = u_n^{(k)}(t_{in}) - \mathscr{L}_n(u_n^{(k)})(t_{in})$, $0 \leq i \leq n$.
   Recall $\mathscr{L}_n(v) = \mathscr{K}_n v - \mathscr{S}_n[\mathscr{G}(v) - f]$.
2. $q_{im} = [\mathscr{L}_n'(u_n^{(k)})r_n](t_{im})$, $0 \leq i \leq m$,
   $q_{in} = [\mathscr{L}_n'(u_n^{(k)})r_n](t_{in})$, $0 \leq i \leq n$.
   Recall $\mathscr{L}_n'(v)r = \mathscr{K}_n r - \mathscr{S}_n[\mathscr{G}'(v)r]$.
3. Solve the linear system $[I - \widehat{\mathscr{L}}_m'(u_m)]\delta_m = q_m$, where $I - \widehat{\mathscr{L}}_m'(u_m)$ is the matrix of order $m+1$ associated with $I - \mathscr{L}_m'(u_m)$. The LU factorization will generally have been computed earlier and saved for use in the present iteration. The vector q_m was defined in step 2.
4. $\delta_{in} = q_{in} + [\mathscr{L}_m'(u_m)\delta_m](t_{in})$,
   $u_n^{(k+1)}(t_{in}) = u_n^{(k)}(t_{in}) - r_n^{(k)}(t_{in}) - \delta_{in}$, $0 \leq i \leq n$.

In the evaluation of the product integration portion of $\mathscr{S}_n v$ at the node points $\{t_{in}\}$ and $\{t_{im}\}$, it is possible to be quite efficient, both in the evaluation of the needed product integration weights and in the size of the tables that need to be constructed in advance of the computation. A partial discussion of the computation of these weights is given in Atkinson [4, p. 113].

This iteration has $O(n^2)$ arithmetic operations per iterate. More precisely, we use the following operations:

(a) $n+1$ evaluations of the function g, used in evaluating $\mathscr{G}(u_n^{(k)})$.
(b) $n+1$ evaluations of the partial derivative g_u, used in evaluating $\mathscr{G}'(u_n^{(k)})$.
(c) $2n(2n+1)$ arithmetic operations to evaluate r_n.
(d) $4n(n+m) + 2n + m$ operations to evaluate q_m and q_n.
(e) $2m^2$ operations to solve for δ_m.
(f) $4nm + 3n + m$ operations to calculate δ_n and $u_n^{(k+1)}$.

It is assumed that the needed matrices associated with $\mathscr{K}_n$, $\mathscr{K}_m$, $\mathscr{S}_n$, and $\mathscr{S}_M$ have all been computed once for use in all iterations.

The total number of arithmetic operations needed in computing one iteration of (3.10) is approximately

$$8n^2 + 8nm + 2m^2 + 7n + 2m.$$

For comparison, the analogous operations count for the two-grid method (3.6) is

$$4n^2 + 8nm + 2m^2 + 5n + 2m.$$

Since the term n^2 will dominate the remaining terms in these operation counts for $n \gg m$, we have that the two-grid method (3.10) is about twice as expensive per iterate as the two-grid method (3.6). Combining this with the generally faster convergence of (3.10), the two methods seem to be roughly comparable in computation time. Nonetheless, we have a slight preference for method (3.10), mainly because of the convergence behavior. With nonlinear integral equations other than those considered here, the first method (3.6) has been less regular in its convergence, with the values of *Ratio* for (3.6) varying widely as the iteration converged; see the discussion of method (3.6) for linear equations in Atkinson [4, pp. 159–161]. But for the present work, the convergence behavior of method (3.6) was very regular, equal to that seen with method (3.10), in contrast with earlier work on other nonlinear integral equations. Thus there does not seem to be any clear reason to prefer either of these two-grid methods over the other one.

4. Trigonometric product integration

For boundary value problems (1.1)–(1.2) with a smooth boundary Γ, we can usually do better than a fixed order of convergence $O(n^{-p})$, $p > 0$, of the type obtained in §2. In this section, this is accomplished by using trigonometric polynomial interpolation to define the product integration approximation for the single layer operator $\mathscr{S}$. This gives a much improved approximation u_n as compared to using piecewise polynomial interpolation.

For $n \geq 1$, let

$$h = \frac{2\pi}{2n+1}, \qquad t_j = jh, \quad j = 0, \pm 1, \pm 2, \ldots.$$

The trigonometric polynomial of degree $\leq n$ that interpolates a given function $\rho \in C_p(2\pi)$ on the nodes $\{t_j\}$ is given by

$$\rho_n(t) = \sum_{-n}^{n} \rho(t_j) l_j(t) \tag{4.1}$$

with the Lagrange functions l_j given by

$$l_j(t) = \frac{h}{\pi} D_n(t - t_j), \qquad D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos(kt) = \frac{\sin(n + \frac{1}{2})t}{2\sin(\frac{1}{2}t)}.$$

Also define the linear operator $P_n \colon C_p(2\pi) \to C_p(2\pi)$ by $P_n\rho = \rho_n$.

To approximate the single layer integral $\mathcal{S}v$ of (2.7), first rewrite it as

$$\begin{aligned} &\mathcal{S}v(t) = \mathcal{S}_0\rho(t) + \mathcal{B}\rho(t), \qquad \rho(t) = v(t)|r'(t)|, \\ &\mathcal{S}_0\rho(t) = -\frac{1}{\pi}\int_0^{2\pi} \rho(s)\log\left|\frac{2}{\sqrt{e}}\sin\frac{1}{2}(t-s)\right|, \\ &\mathcal{B}\rho(t) = -\frac{1}{\pi}\int_0^{2\pi} B(t,s)\rho(s)\,ds, \\ &B(t,s) = \begin{cases} \log\left|\dfrac{\sqrt{e}[r(t)-r(s)]}{2\sin\frac{1}{2}(t-s)}\right|, & t-s \neq 2k\pi, \\ \log|\sqrt{e}r'(t)|, & t-s = 2k\pi, \end{cases} \end{aligned} \tag{4.2}$$

with k any integer. With respect to both variables t and s, the function B belongs to $C_p^{l-1}(2\pi)$ if r belongs to $C_p^l(2\pi)$, some $l \geq 1$. For similar splittings of $\mathcal{S}$, see Kress [7] and the references contained therein.

This decomposition is taken from Yan and Sloan [9]. The operator $\mathcal{S}_0$ is closely related to the single layer operator $\mathcal{S}$ on the unit circle. It satisfies

$$\mathcal{S}_0 \colon H^t(2\pi) \xrightarrow[\text{onto}]{\text{1-1}} H^{t+1}(2\pi), \qquad t \geq 0. \tag{4.3}$$

The eigenfunctions of $\mathcal{S}_0$ are the trigonometric monomials:

$$\mathcal{S}_0 \colon e^{ikt} \to \frac{1}{\operatorname{Max}\{1, |k|\}} e^{ikt} \tag{4.4}$$

for all integers k.

Approximate $\mathcal{S}_0\rho$ by replacing ρ by its trigonometric interpolant ρ_n of (4.1), and then perform the integrations exactly. This yields

$$\mathcal{S}_{0,n}\rho(t) = \mathcal{S}_0 P_n\rho(t) = \sum_{j=-n}^{n} \rho(t_j)E_n(t-t_j) \tag{4.5}$$

with

$$E_n(t) = \frac{h}{\pi}\left\{\frac{1}{2} + \sum_{k=1}^{n}\frac{1}{k}\cos(kt)\right\}. \tag{4.6}$$

Approximate $\mathcal{S}v$ by

$$\mathcal{S}_n v = \mathcal{S}_{0,n}\rho + \mathcal{B}_n\rho, \qquad \rho(t) = v(t)|r'(t)|, \tag{4.7}$$

where $\mathcal{B}_n\rho$ is obtained from $\mathcal{B}\rho$ by applying the trapezoidal rule. It is possible to evaluate $E_n(t)$ very efficiently for larger values of n. But we found that $n \leq 20$ was sufficient for most of our examples; and therefore, we used (4.6) directly, evaluating $\{\cos(kt) | 1 \leq k \leq n\}$ with the trigonometric addition formulas. Thus, the cost of evaluating $E_n(t)$ was about $8n$ arithmetic operations plus the cost of evaluating $\sin t$ and $\cos t$.

Approximate the nonlinear integral equation (2.4) by

$$u_n - \mathcal{K}_n u_n + \mathcal{S}_n\mathcal{G}(u_n) = \mathcal{S}_n f. \tag{4.8}$$

The trapezoidal rule applied to $\mathcal{K}u$ is used to define the numerical integral operator $\mathcal{K}_n$. As before in §2, this approximating equation is a Nyström method, and the solving of (4.8) reduces to solving a finite system of nonlinear equations of order $2n+1$.

The error analysis of (4.8) is similar to that used in Theorem 1, but it requires some significant changes. With (4.8), we cannot prove $\{\mathcal{S}_n\}$ is a collectively compact and pointwise convergent family of operators from $C_p(2\pi)$ into $C_p(2\pi)$; and therefore, we cannot simply invoke the results from [2] in the manner done in Theorem 1. Instead, we construct another proof, for our situation, of the major result [2, Theorem 2]. Then we can use the subsequent results in [2], constructing an error analysis for (4.8) in analogy with that given earlier in Theorems 1–3 for the product quadratic method (2.10). We begin with the following lemma on trigonometric interpolation. It can be proven using a fairly standard manipulation of the Fourier series representation of the function ρ.

Lemma 4. *Let $\rho \in H^t(2\pi)$ with $t > \frac{1}{2}$; let $s < t$. Then the trigonometric interpolation polynomial $P_n\rho$ of* (4.1) *satisfies*

$$\|\rho - P_n\rho\|_s \le \frac{c_s\|\rho\|_t}{n^{t-s}} \tag{4.9}$$

with c_s a constant and $\|\cdot\|_t$ denoting the norm in $H^t(2\pi)$.

Using this lemma, we can prove a number of error bounds involving the approximating single layer $\mathcal{S}_{0,n}$.

Lemma 5. *Let $\rho \in H^t(2\pi)$ with $t > \frac{1}{2}$; let $s < t+1$. Then*

$$\|\mathcal{S}_0\rho - \mathcal{S}_{0,n}\rho\|_s \le \frac{c_s\|\rho\|_t}{n^{t-s+1}}. \tag{4.10}$$

Proof. Use (4.3) to obtain

$$\|\mathcal{S}_0\rho - \mathcal{S}_{0,n}\rho\|_s = \|\mathcal{S}_0[I - P_n]\rho\|_s \le \|\mathcal{S}_0\|\,\|(I-P_n)\rho\|_{s-1}.$$

Then invoke Lemma 4 to complete the proof. □

Lemma 6. *For any $\varepsilon > 0$,*

$$\|(\mathcal{S}_0 - \mathcal{S}_{0,n})\mathcal{S}_0\| \le \frac{c_\varepsilon}{n^{1.5-\varepsilon}}, \tag{4.11}$$

$$\|(\mathcal{S}_0 - \mathcal{S}_{0,n})\mathcal{S}_{0,n}\| \le \frac{c_\varepsilon \log n}{n^{1.5-\varepsilon}}, \tag{4.12}$$

where the operators $\mathcal{S}_0$ and $\mathcal{S}_{0,n}$ are regarded as operators from $C_p(2\pi)$ into $C_p(2\pi)$. The constant c_ε depends only on ε.

Proof. We show only (4.12), the more difficult of the two inequalities. The proof of (4.11) is similar. For $\rho \in C_p(2\pi) \subset H^0(2\pi)$, we have $\mathcal{S}_0\rho \in H^1(2\pi)$.

Similarly, $P_n\rho \in C_p(2\pi)$ and $\mathscr{S}_{0,n}\rho \in H^1(2\pi)$. Using the Sobolev embedding theorem, it follows that for any $\varepsilon > 0$,

$$\begin{aligned}\|(\mathscr{S}_0 - \mathscr{S}_{0,n})\mathscr{S}_{0,n}\rho\|_\infty &\le \|(\mathscr{S}_0 - \mathscr{S}_{0,n})\mathscr{S}_{0,n}\rho\|_{1/2+\varepsilon} \\ &\le \frac{c_\varepsilon}{n^{1.5-\varepsilon}}\|\mathscr{S}_0 P_n\rho\|_1 \le \frac{c_\varepsilon}{n^{1.5-\varepsilon}}\|P_n\rho\|_0 \\ &\le \frac{c_\varepsilon}{n^{1.5-\varepsilon}}\|P_n\rho\|_\infty \le \frac{c_\varepsilon \log n}{n^{1.5-\varepsilon}}\|\rho\|_\infty .\end{aligned}$$

The second inequality uses Lemma 5 with $s = \frac{1}{2} + \varepsilon$, and the third inequality uses (4.3). The last inequality uses the result

$$\|P_n\| = O(\log n) \tag{4.13}$$

when P_n is regarded as an operator from $C_p(2\pi)$ into itself; see Zygmund [10, Vol. II, p. 18]. The constant c_ε is regarded as generic. □

We make a further assumption about the function g and the associated operator $\mathscr{G}$. Assume:

[A4] For some $p > \frac{1}{2}$, $\mathscr{G}\colon H^p(2\pi) \to H^p(2\pi)$ is bounded with

$$\|\mathscr{G}(\alpha v)\|_p \le c\alpha^q \|\mathscr{G}(v)\|_p, \qquad v \in H^p(2\pi),\ |\alpha| \ge \alpha_0, \tag{4.14}$$

for some $q \ge 0$ and some $\alpha_0 > 0$. c is a constant independent of v and α.

This is easily verified in most cases. Consider the earlier examples g in (2.21)–(2.22):

(a) For $g = g_1$, [A4] is true for any $p \ge 0$ with $q = 0$.
(b) For $g = g_2$, [A4] is true for $0 \le p < 4$ with finite q depending on p.

The assumption [A4] can be weakened a great deal, but it is needed for the proof we give of the following lemma. The assumption [A4] can be forced to be satisfied in most cases, as follows: Replace $g(P, v)$ by $\tilde{g}(P, v) \equiv \psi(v)g(P, v)$, where (a) $\psi(v) = 1$ for all real numbers v in a neighborhood of $F = \{g(P, u_0(P)) | P \in \Gamma\}$, say within a distance of $d_0 > 0$, with u_0 the true solution of (2.4); (b) $\psi(v) = 0$ for all real numbers v well away from F, say at a distance greater than $d_1 > d_0$; and (c) $\psi \in C^\infty(\mathbb{R})$. Then the problem (1.1)–(1.2) with g replaced with $\tilde{g}$ will have the same solution u_0, and the new operator

$$\mathscr{L}\colon v \to \mathscr{L}(v) \equiv \mathscr{K}v - \mathscr{S}\mathscr{G}(v) + \mathscr{S}f$$

will have the same values in $C_p(2\pi)$ for functions v near to u_0. Finally, the new equation will satisfy (4.14) for all sufficiently large values of α with $q = 0$.

For notational convenience, we will also write the approximating equation (4.8) as

$$u_n = \mathscr{L}_n(u_n),$$

as was done in earlier sections. The following lemma is crucial to proving [2, Theorem 2] for our situation.

Lemma 7. *Assume* [A1] *and* [A4]. *Let* T *be a bounded set in* $C_p(2\pi)$. *Then*

$$\sup_{v\in T}\|(\mathscr{S}-\mathscr{S}_n)\mathscr{G}(\mathscr{L}_n(v))\|_\infty \to 0 \quad \textit{as } n\to\infty. \tag{4.15}$$

Proof. From the assumption that Γ is C^∞, it follows that the kernel functions $K(t, s)$ and $B(t, s)$ are also C^∞ in both variables t and s, as well as being periodic. This implies $\mathscr{B}$, $\mathscr{B}_n$, $\mathscr{K}$, and $\mathscr{K}_n$ are bounded mappings from $C_p(2\pi)$ into $C_p^m(2\pi)$, for all integers $m \geq 0$.

We write

$$\begin{aligned}(\mathscr{S}-\mathscr{S}_n)\mathscr{G}(\mathscr{L}_n(v)) &= (\mathscr{S}_0-\mathscr{S}_{0,n})\widehat{\mathscr{G}}(\mathscr{L}_n(v))\\ &\quad + (\mathscr{B}-\mathscr{B}_n)\widehat{\mathscr{G}}(\mathscr{L}_n(v)),\end{aligned} \tag{4.16}$$

where

$$\widehat{\mathscr{G}}(v)(t) \equiv |r'(t)|\mathscr{G}(v)(t).$$

We first show

$$\sup_{v\in T}\|(\mathscr{S}_0-\mathscr{S}_{0,n})\widehat{\mathscr{G}}(\mathscr{L}_n(v))\|_\infty \to 0 \quad \text{as } n\to\infty. \tag{4.17}$$

The same result for the last term in (4.16) follows by a similar, but simpler, argument.

Combine Lemma 5, the Sobolev embedding theorem, and [A4] to obtain

$$\|(\mathscr{S}_0-\mathscr{S}_{0,n})\widehat{\mathscr{G}}(\mathscr{L}_n(v))\|_\infty \leq \frac{c}{n^{p+.5-\varepsilon}}\|\widehat{\mathscr{G}}(\mathscr{L}_n(v))\|_p \tag{4.18}$$

for the index p of [A4]. We now show

$$\|\widehat{\mathscr{G}}(\mathscr{L}_n(v))\|_p \leq c[\log n]^q. \tag{4.19}$$

The constant c is generic in this proof.

Write

$$\mathscr{L}_n(v) = \mathscr{K}_n v - \mathscr{S}_0 P_n\widehat{\mathscr{G}}(v) - \mathscr{B}_n\widehat{\mathscr{G}}(v) + \mathscr{S}_n f.$$

From the smoothing behavior of $\mathscr{S}_0$, $\mathscr{K}_n$, and $\mathscr{B}_n$, it is straightforward to show

$$\|\mathscr{L}_n(v)\|_p \leq c_1\|v\|_\infty + c_2\{\|P_n\|+1\}\|\widehat{\mathscr{G}}(v)\|_\infty + c_3\|P_n\|, \qquad v\in T.$$

The set $\widehat{\mathscr{G}}(T)$ is easily bounded from [A1]. Using (4.13),

$$\sup_{v\in T}\|\mathscr{L}_n(v)\|_p \leq c\log n. \tag{4.20}$$

Write

$$\widehat{\mathscr{G}}(\mathscr{L}_n(v)) = \widehat{\mathscr{G}}(\alpha_n w_n), \qquad \alpha_n = \log n,\ w_n = \frac{1}{\log n}\mathscr{L}_n(v).$$

Define

$$\widehat{T} = \{w_n | v\in T \text{ and } n\geq 1\}.$$

From (4.20), $\widehat{T}$ is bounded in $H^p(2\pi)$. From [A1] and [A4],

$$\|\widehat{\mathscr{G}}(\mathscr{L}_n(v))\|_p \leq c\alpha_n^q\|\widehat{\mathscr{G}}(w_n)\|_p \leq c[\log n]^q,$$

as asserted in (4.19).

Combining (4.19) and (4.18),

$$\|(\mathscr{S}_0 - \mathscr{S}_{0,n})\widehat{\mathscr{G}}(\mathscr{L}_n(v))\|_\infty \le \frac{c[\log n]^q}{n^{p+.5-\varepsilon}}. \tag{4.21}$$

This easily converges to zero as $n \to \infty$. The proof can be easily generalized to the last term in (4.16), and that will complete the proof of (4.15). □

The result (4.15) and its proof can also be extended to show

$$\begin{aligned}
&\operatorname*{Sup}_{v\in T}\|(\mathscr{S} - \mathscr{S}_n)\mathscr{G}(\mathscr{L}(v))\|_\infty \to 0,\\
&\operatorname*{Sup}_{v\in T}\|(\mathscr{K} - \mathscr{K}_n)\mathscr{G}(\mathscr{L}_n(v))\|_\infty \to 0,\\
&\operatorname*{Sup}_{v\in T}\|(\mathscr{K} - \mathscr{K}_n)\mathscr{G}(\mathscr{L}(v))\|_\infty \to 0
\end{aligned}$$

as $n \to \infty$. The proofs are much the same, with (4.15) being the most difficult of the lot.

With these results, the proof of Theorem 2 of [2] can be completed with no difficulty. Then the other results of [2] can be applied to our approximation (4.8). Theorems 1 and 2 of §2 are true for (4.8); but we omit their statement since it is exactly the same as previously given. We give a convergence result, using Lemma 5 and the following assumption.

[A5] For some integer $\lambda > 0$, the function g satisfies

$$u \in C_p^\lambda(2\pi) \quad \text{implies} \quad g(\cdot, u(\cdot)) \in C_p^\lambda(2\pi).$$

Theorem 8. *Assume* [A1], [A2], [A4], [A5], *and the remaining hypotheses of Theorem* 2. *Further, assume the solution* u_0 *of* (2.4) *belongs to* $C_p^\lambda(2\pi)$. *Then*

$$\|u_0 - u_n\|_\infty \le c/n^\lambda \tag{4.22}$$

for n *sufficiently large.* [Remark: The assumption [A5] can be replaced by the weaker hypothesis that both u_0 and $g(\cdot, u_0(\cdot))$ belong to C_p^λ.]

Proof. Use the inequality (2.19) of Theorem 2 to write

$$\begin{aligned}
\|u_0 - u_n\|_\infty \le \gamma\{\|(\mathscr{K} - \mathscr{K}_n)u_0\|_\infty + \|\mathscr{S}_0(I - P_n)\widehat{\mathscr{G}}(u_0)\|_\infty \\
+ \|(\mathscr{B} - \mathscr{B}_n)\mathscr{G}(u_0)\|_\infty\}.
\end{aligned}$$

The terms $\|(\mathscr{K} - \mathscr{K}_n)u_0\|_\infty$ and $\|(\mathscr{B} - \mathscr{B}_n)\mathscr{G}(u_0)\|_\infty$ can be treated with standard error results for the trapezoidal rule, yielding the order of convergence $n^{-\lambda}$. The term $\|\mathscr{S}_0(I - P_n)\widehat{\mathscr{G}}(u_0)\|_\infty$ can be bounded with the Sobolev embedding theorem and Lemma 5, giving an order of convergence of approximately $n^{-\lambda-.5}$. This proves (4.22). □

TABLE 7
Using method (4.8) *to solve* (1.4) *with* $g = g_1$

a	b	n	$\|u_0 - u_n\|_\infty$	IT	$COND$
1.0	2.0	3	2.23E − 1	5	5.16
		6	1.21E − 3	6	4.75
		9	2.77E − 6	6	5.26
		12	3.53E − 9	6	5.36
		15	2.66E − 12	6	5.21
1.0	5.0	5	4.59E − 1	6	16.5
		10	1.74E − 3	6	15.0
		15	3.06E − 5	6	16.6
		20	5.57E − 7	6	17.6
		25	9.79E − 9	6	17.9
		30	1.70E − 10	6	18.5
1.0	8.0	30	1.75E − 6	6	35.6

TABLE 8
Using method (4.8) *to solve* (1.4) *with* $g = g_2$

a	b	n	$\|u_0 - u_n\|_\infty$	IT	$COND$
1.0	2.0	3	1.45E − 1	9	780
		6	8.27E − 4	9	631
		9	1.82E − 6	9	609
		12	3.59E − 9	9	643
		15	2.38E − 12	9	626

Examples. Consider again the cases (2.21), (2.22) of g, which were used in the examples of §§2 and 3. The equation (4.8) reduces to a finite system of $2n + 1$ nonlinear equations. This system was solved with Newton's method, with the initial guess

$$u_n^{(0)} = u_0 + 1, \qquad P \in \Gamma.$$

The numerical results for various (a, b) and n are given in Tables 7 and 8. For the meaning of IT and *COND*, see the discussion for Tables 1 and 2.

The method (4.8) converges very rapidly as n increases, and it is much more rapid in convergence than the quadratic product integration method of §2. It can be seen from the tables that the rate of convergence is exponential for both problems being solved. Using method (4.8), more elongated and difficult boundaries Γ can be treated very accurately with much smaller values of n than is possible with the quadratic product integration method.

CONCLUSIONS

Although the examples in this paper were all computed on elliptical regions, we did similar calculations on other regions with a smooth boundary.

The numerical results were comparable to those given here, including the much greater accuracy obtained with the trigonometric product integration method of §4. We conclude that when the boundary curve is smooth, as in this paper, then the method of §4 is much superior to the piecewise polynomial product integration method of §2. If the method of §2 is to be used, then the two-grid methods of §3 are a very efficient way to solve the nonlinear systems that arise in the method, and they are preferable to either the ordinary Newton method or the modified Newton method.

In future papers, the ideas of §§2 and 3 will be extended to planar problems on regions whose boundary is only piecewise smooth.

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