SIAM J. NUMER. ANAL.
Vol. 32, No. 3, pp. 969–983, June 1995

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TWO-DIMENSIONAL QUADRATURE FOR FUNCTIONS WITH A POINT SINGULARITY ON A TRIANGULAR REGION *

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Abstract. We consider the numerical integration of functions with point singularities over a planar wedge S using isoparametric piecewise polynomial interpolation of the function and the wedge. Such integrals often occur in solving boundary integral equations using the collocation method. To obtain the same order of convergence as is true with uniform meshes for smooth functions, we introduce an adaptive refinement of the triangulation of S. Error analyses and several examples are given for a certain type of adaptive refinement.

Key words. numerical integration, quadratic interpolation, adaptive refinement

AMS subject classifications. 65D30, 65D32

1. Introduction. We consider the problem of approximating an integral of the form

$$\int_S f(x,y)\,dS$$

where S is a closed, bounded, connected set in $\mathbf{R}^2$. When the integrand has singularities within the integration region, the use of a standard quadrature method may be very inefficient. There are several ways to deal with this problem. One approach is to use a change of variables [10] to transform a singular integrand into a well-behaved function over a new region. A second approach is to use adaptive numerical integration to place more node points near where the integrand is badly behaved to improve the performance of a standard quadrature method. A third approach is to use extrapolation methods to construct new and more accurate integration formulas based on the asymptotic expansion for the quadrature error in the original quadrature rule. The generalization of the classical Euler–Maclaurin expansion to functions having a particular type of singularity, as obtained by Lyness [8], [9], provides a basis for extrapolation methods in the same way as the Euler–Maclaurin expansion is used as a basis for Romberg integration. For the one-dimensional case, a standard discussion of these methods and others can be found in Atkinson [3].

In this paper we discuss adaptive numerical integration for the two-dimensional case. The process of placing node points with variable spacing so as to better reflect the integrand is called adaptive refinement or grading the mesh. We propose a type of adaptive refinement for which the order of convergence is the same as for smooth functions, but with the integrand function having a particular type of singularity, specified below.

In the case of smooth integrand functions, Chien [6], [7] obtained that for integrals over a piecewise smooth surface in $\mathbf{R}^3$, the numerical integration using isoparametric piecewise quadratic interpolation for both the surface and the integrand leads to the

* Received by the editors April 12, 1993; accepted for publication (in revised form) December 21, 1993.

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order of convergence $O(\delta^4)$. In this, $\delta = \max_{1 \le K \le N}(\delta_K), \delta_K = \max_{p,q \in \Delta_K} |p-q|$, and $\{\Delta_K | K = 1, \dots, N\}$ is a quasi-uniform triangulation of the surface. In terms of the number N of triangles, the order of convergence is $O(1/N^2)$. We extend these results to singular integrands.

To simplify our discussion, we study only the following class of problems. The integration region S is a wedge, i.e.,

$$S = \{(x,y) \in \mathbf{R}^2 | 0 \le r \le 1, 0 \le \theta \le \Theta\}, \qquad 0 < \Theta < \pi.$$

The integration function is singular at only the origin, and it is of the form

$$f(x,y) = r^\alpha \varphi(\theta) h(r) g(x,y), \qquad \alpha > -2 \tag{1.1}$$

or the form

$$f(x,y) = r^\alpha \ln r \varphi(\theta) h(r) g(x,y), \qquad \alpha > -2. \tag{1.2}$$

Here (x,y) are the Cartesian coordinates with the corresponding polar coordinates (r,θ). The functions $\varphi(\theta), h(r), g(x,y)$ are assumed to be sufficiently smooth. This is also the class of functions considered in Lyness [8].

In §2, we describe the triangulation of the wedge S and the adaptive refinement scheme we use. The interpolation-based quadrature formulas are given in §3. Section 4 contains the error analyses and a discussion of the results. Numerical examples are given in §5.

This paper presents detailed results for only the use of quadratic interpolation. The method being used generalizes to other degrees of piecewise polynomial interpolation, and the results are consistent with the kind of results we have obtained for the quadratic case. This generalization for other degrees of interpolation is given in §6.

Because integrals of nonsmooth functions often occur in solving boundary integral equations using the numerical method, we expect that the error analysis of the present numerical integration methods eventually will lead to better numerical methods for the solution of boundary integral equations.

2. The triangulation and adaptive refinement. In this section, we describe the triangulation of the wedge S and discuss its refinement to a finer mesh. Let

$$\tau_n = \{\Delta_{K,n} | 1 \le K \le N_n\} \tag{2.1}$$

be the triangulation mesh for a sequence $n = 1, 2, \dots$. When referring to the element $\Delta_{K,n}$, the reference to n will be omitted, but understood implicitly.

The initial triangulation τ_1 of S is obtained by connecting the midpoints of the sides of S using straight lines. The sequence of triangulations τ_n of (2.1) will be obtained by successive adaptive and uniform refinements based on the initial triangulation. We construct this sequence as follows, and we call it an $La + u$ refinement with L a positive integer. Given $\{\Delta_{K,n} | 1 \le K \le N_n\}$, we divide the triangle containing the origin into four new triangular elements. For the resulting triangulation, repeat the preceding subdivision. After doing this L times, we divide simultaneously every triangle into four new triangles. The final triangulation is denoted by $\{\Delta_{K,n+1} | 1 \le K \le N_{n+1}\}$. In other words, at level n, we perform L times an adaptive subdivision on the triangle containing the origin, and then we do one simultaneous

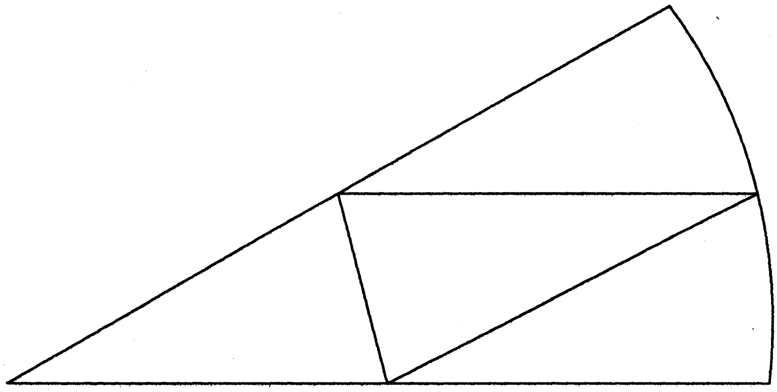

FIG. 1. $n = 1$.

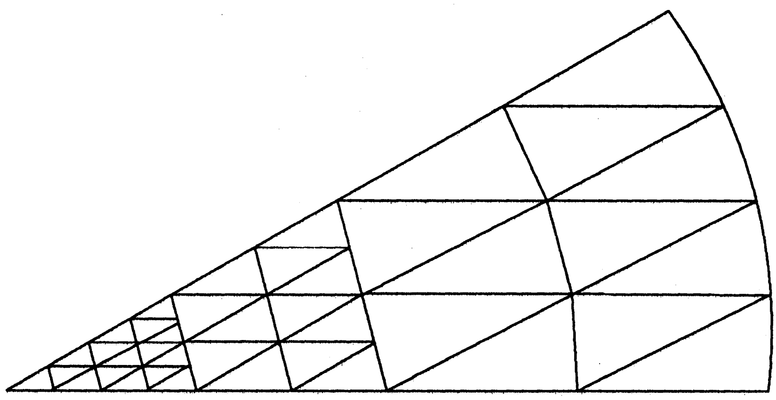

FIG. 2. $n = 2$.

subdivision of all triangles. An advantage of this form of refinement is that each set of mesh points contains those mesh points at the preceding level.

As an example, we illustrate the $2a + u$ refinement for $n = 1, 2$ in Figs. 1 and 2. When $n = 2$, there are three different-sized triangular elements. Let $\eta = \frac{\sin\Theta}{1-\cos\Theta}$, and define

$$B_0 = \left\{(x, y) \in S | \eta x + y \geq \frac{\eta}{2}\right\},$$

$$B_1 = \left\{(x, y) \in S | \eta/4 \leq \eta x + y \leq \frac{\eta}{2}\right\},$$

$$B_2 = \left\{(x, y) \in S | 0 \leq \eta x + y \leq \frac{\eta}{4}\right\}.$$

The set B_l is the union of the triangles of the same size in S, and therefore it is uniformly divided by the triangulation. The diameter of triangles in B_l is $O(2^{-(2+l)})$. Moreover, functions in (1.1) and (1.2) are smooth on $B_0 \cup B_1$.

More generally, by examining the structure of the $La + u$ refinement, we can calculate the total number N_n of triangles at level n: this is $(L+1)2^{2n} - 4L = O(2^{2n})$. There are $Ln - (L - 1)$ different-sized triangular elements. The closer the triangle is to the origin, the smaller it is. As the triangles vary in size from large to small, we name

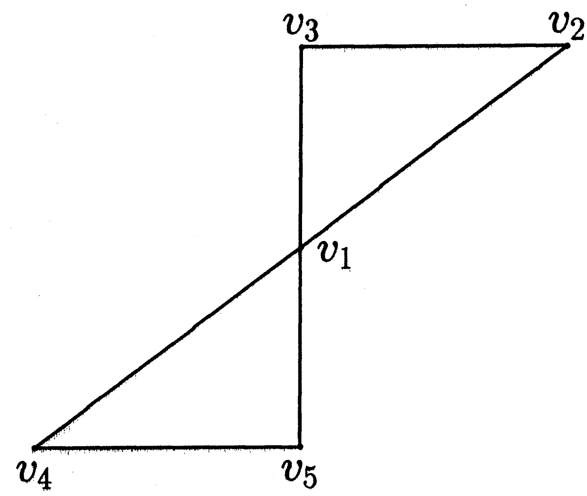

FIG. 3. *A symmetric pair of triangles.*

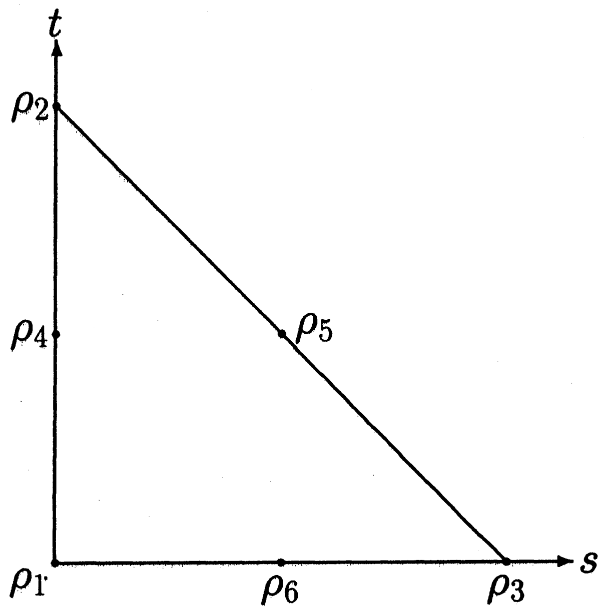

FIG. 4. *The unit simplex.*

the region containing triangles of the same size to be $B_0, B_1, \ldots, B_{Ln-L}$, respectively. The diameter of triangles in B_l, denoted by δ_l, is $O(2^{-(n+l)})$. Let N_l be the number of triangles in B_l. Then N_l is proportional to 4^{n-i} where $l = iL + i_1$ for $0 \le i_1 \le L-1$. The distance from the origin to B_l, denoted by r_l, is $O(1/(2^{(L+1)i}))$.

In each $B_l(l = 1, \ldots, Ln - L)$, the triangular elements Δ_K are true triangles and all are congruent. The triangular elements in B_0 are nearly congruent. More importantly, any *symmetric pair of triangles,* as shown in Fig. 3, have the following properties:

$$v_1 - v_2 = -(v_1 - v_4),$$
$$v_1 - v_3 = -(v_1 - v_5).$$

The total number of symmetric pairs of triangles in B_l is $O(N_l)$, and the remaining triangles in B_l is $O(\sqrt{N_l})$.

If $L = 0$, then the refinement is quasi uniform. The analysis given in [6] indicates that a quasi-uniform refinement is a better scheme to use with smooth integrands.

3. Interpolation. Let σ denote the unit simplex in the *st-plane*

$$\sigma = \{(s,t) | 0 \le s, t, s+t \le 1\}.$$

Let $\rho_1, \dots, \rho_6$ denote the three vertices and three midpoints of the sides of σ, which are numbered according to Fig. 4.

To define interpolation, introduce the basis functions for quadratic interpolation on σ. Letting $u = 1 - (s+t)$, we define

$$\begin{aligned} &l_1(s,t) = u(2u-1), \quad && l_2(s,t) = t(2t-1), \quad && l_3(s,t) = s(2s-1), \\ &l_4(s,t) = 4tu, && l_5(s,t) = 4st, && l_6(s,t) = 4su. \end{aligned}$$

We give the corresponding set of basis functions $\{l_{j,K}(q)\}$ on Δ_K by using its parameterization over σ. As a special case of piecewise smooth surfaces in $\mathbf{R}^3$, discussed in [5]–[7], there is a mapping

$$m_K : \sigma \underset{\text{onto}}{\xrightarrow{1-1}} \Delta_K \tag{3.1}$$

with $m_K \in C^6(\sigma)$. Introduce the node points for Δ_K by $v_{j,K} = m_K(\rho_j), j = 1, \dots, 6$. The first three are the vertices and the last three are the midpoints of the sides of Δ_K. Define

$$l_{j,K}(m_K(s,t)) = l_j(s,t), \quad 1 \le j \le 6, \quad 1 \le K \le N.$$

Given a function f, define

$$\mathcal{P}_N f(q) = \sum_{j=1}^{6} f(v_{j,K}) l_{j,K}(q), \qquad q \in \Delta_K$$

for $K = 1, \dots, N$. This is called the *piecewise quadratic isoparametric function interpolating* f on the nodes of the mesh $\{\Delta_K\}$ for S.

4. Numerical integration and error analyses. With the triangulation $\{\Delta_K\}$ and the mapping $m_K : \sigma \underset{\text{onto}}{\xrightarrow{1-1}} \Delta_K$, we have

$$\int_{\Delta_K} f(q)\, dS = \int_\sigma f(m_K(s,t)) |D_s m_K(s,t) \times D_t m_K(s,t)|\, ds\, dt. \tag{4.1}$$

D_s and D_t denote differentiation with respect to s and t, respectively. The quantity $|D_s m_K(s,t) \times D_t m_K(s,t)|$ is the Jacobian determinant of the mapping $m_K(s,t)$ used in transforming surface integrals over Δ_K into integrals over σ. When Δ_K is a triangle, the Jacobian is twice the area of Δ_K.

The numerical integration formula used here is

$$\int_{\sigma} g(s,t)\,ds\,dt \approx \frac{1}{6}[g(\rho_4)+g(\rho_5)+g(\rho_6)], \tag{4.2}$$

which is based on integrating the quadratic polynomial interpolating g on σ at $\rho_1,\dots,\rho_6$. This integration has degree of precision 2. Applying (4.2) to the right side of (4.1), we have

$$\int_{\Delta_K} f(q)\,dS \approx \frac{1}{6}\sum_{j=4}^{6} f(m_K(\rho_j))|D_s m_K(s,t)\times D_t m_K(s,t)|_{\rho_j}. \tag{4.3}$$

A major problem with (4.3) is that $D_s m_K$ and $D_t m_K$ are inconvenient to compute for some elements Δ_K on many surfaces S. Therefore, we approximate $m_K(s,t)$ in terms of only its values at $\rho_1,\dots,\rho_6$. Define

$$\tilde{m}_K(s,t) = \sum_{j=1}^{6} m_K(\rho_j) l_j(s,t) = \sum_{j=1}^{6} v_{j,K} l_j(s,t).$$

For the case of the wedge of a circle, in this paper only the outer triangles will be affected by this approximation. Then

$$\begin{aligned}\int_{\Delta_K} f(q)\,dS &\approx \frac{1}{6}\sum_{j=4}^{6} f(m_K(\rho_j))|D_s\tilde{m}_K(s,t)\times D_t\tilde{m}_K(s,t)|_{\rho_j}\\ &= \sum_{j=4}^{6}\omega_{j,K} f(v_{j,K})\end{aligned}$$

where

$$\omega_{j,K} = \frac{1}{6}|D_s\tilde{m}_K(s,t)\times D_t\tilde{m}_K(s,t)|_{\rho_j}.$$

LEMMA 1. *Let $\{\Delta_K\}$ be the irregular triangulation defined by the $La+u$ refinement, and let N be its total number of triangles. Assume that f is of the form* (1.1). *Then*

$$\left|\int_{B_{Ln-L}} f(q)\,dS - \sum_{\Delta_K\subset B_{Ln-L}}\sum_{j=4}^{6}\omega_{j,K} f(v_{j,K})\right| \le O\left(\frac{1}{N^{p_1}}\right) \tag{4.4}$$

where $p_1 = \frac{(L+1)(\alpha+2)}{2}$.

Proof. Let

$$\delta_{Ln-L} = \frac{1}{2^{n+(Ln-L)}}.$$

The partition of B_{Ln-L} is shown in Fig. 5. Let $\{\Delta_K | K = 1,\dots,16\}$ denote the sixteen triangles in B_{Ln-L}, and let Δ_1 contain the origin. Then the area of Δ_K is $\delta^2_{Ln-L}\sin(\Theta/2)$.

(a) We first estimate the error of

$$\left|\int_{\Delta_1} f(q)\,dS - \sum_{j=4}^{6}\omega_{j,1} f(v_{j,1})\right|.$$

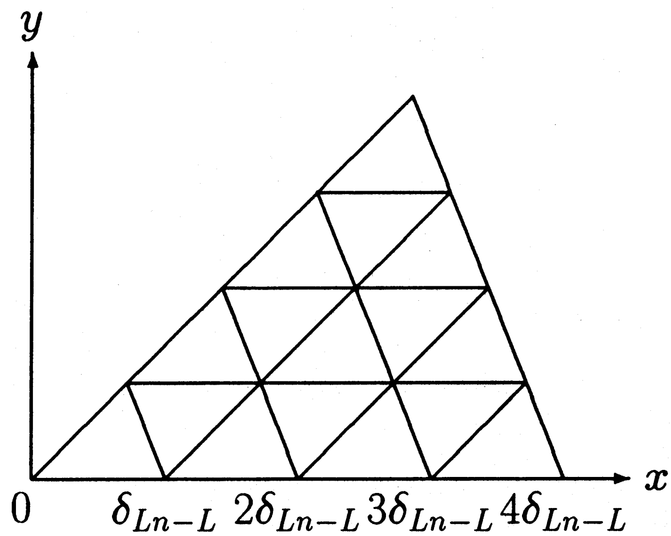

FIG. 5. *The partition of* B_{Ln-L}.

Define

$$g_1(x, y) = \varphi(\theta)h(r)g(x, y),$$

and note that $g_1(x, y)$ is integrable over Δ_1. Since $r^\alpha \geq 0$, by the integral mean value theorem we have

$$\int_{\Delta_1} f(q)\, dS = \mu \int_{\Delta_1} r^\alpha\, dS$$

where

$$\inf_{\Delta_1} g_1(x, y) \leq \mu \leq \sup_{\Delta_1} g_1(x, y).$$

Also, μ is bounded:

$$\begin{aligned} |\mu| &\leq \sup_S |g_1(x, y)| \\ &\leq \sup_S |\varphi(\theta)| \sup_S |h(r)| \sup_S |g(x, y)| \\ &= \max_{0\leq\theta\leq\Theta} |\varphi(\theta)| \max_{0\leq r\leq 1} |h(r)| \max_S |g(x, y)| \\ &\equiv M < \infty. \end{aligned}$$

Therefore,

$$\begin{aligned} \left| \int_{\Delta_1} f(q)\, dS \right| &\leq M \int_{\Delta_1} r^\alpha\, dS \\ &= M \int_0^{\delta_{Ln-L}} \int_0^{\Theta} r^{\alpha+1}\, d\theta\, dr \\ &= \frac{M\Theta}{\alpha+2} \delta_{Ln-L}^{\alpha+2}. \end{aligned}$$

On the other hand,

$$\left| \sum_{j=4}^{6} \omega_{j,1} f(v_{j,1}) \right| \leq \frac{1}{6} \delta_{Ln-L}^{\alpha+2} \sin\Theta \left(2^{1-\alpha} + \left| \cos\left(\frac{\Theta}{2}\right) \right|^\alpha \right) M.$$

This inequality comes from the following:

$$\begin{aligned}\omega_{j,1} &= \frac{1}{6}|D_s\tilde{m}_1(s,t) \times D_t\tilde{m}_1(s,t)|_{\rho_j} \\ &= \frac{1}{6}2(\text{Area of } \Delta_1) \\ &= \frac{1}{6}\delta_{Ln-L}^{\alpha+2}\sin\Theta \quad \text{for } j = 4,5,6\end{aligned}$$

and

$$\begin{aligned}|f(v_{j,1})| &\le |v_{j,1}|^\alpha M \\ &= \begin{cases} \left(\frac{\delta_{Ln-L}}{2}\right)^\alpha M & \text{if } j = 4,6, \\ \delta_{Ln-L}^\alpha \left|\cos\left(\frac{\Theta}{2}\right)\right|^\alpha M & \text{if } j = 5.\end{cases}\end{aligned}$$

Hence,

$$\left|\int_{\Delta_1} f(q)\,dS - \sum_{j=4}^{6}\omega_{j,1}f(v_{j,1})\right| \le O(\delta_{Ln-L}^{\alpha+2}).$$

Notice that $\delta_{Ln-L} = O\left(\frac{1}{N^{(L+1)/2}}\right)$. It follows that the error over Δ_1 is $O\left(\frac{1}{N^{p_1}}\right)$.

(b) The error over $\Delta_K(K = 2,\dots,16)$ can be obtained by using Taylor's error formula. Since f is smooth on Δ_K and $m_K : \sigma \xrightarrow[\text{onto}]{1-1} \Delta_K$ is also smooth, we have

$$f(m_K(s,t)) - \sum_{j=1}^{6} f(m_K(\rho_j))l_j(s,t) = H_{f,K}(s,t;\zeta,\eta) \tag{4.5}$$

where

$$\begin{aligned}H_{f,K}(s,t;\zeta,\eta) = \frac{1}{3!}\Bigg[&\left(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}\right)^3 f(m_K(\zeta,\eta)) \\ &-\sum_{j=1}^{6}\left(s_j\frac{\partial}{\partial s} + t_j\frac{\partial}{\partial t}\right)^3 f(m_K(\zeta_j,\eta_j))l_j(s,t)\Bigg].\end{aligned}$$

In this, $\rho_j = (s_j,t_j)$, (ζ,η) is on the line segment from $(0,0)$ to (s,t), and (ζ_j,η_j) is on the line segment from $(0,0)$ to (s_j,t_j). Notice that (ζ,η) and (ζ_j,η_j) belong to σ. For $(s,t)\in\sigma$, we have

$$0 < \cos\left(\frac{\Theta}{2}\right)\delta_{Ln-L} \le r(m_K(s,t)) \le 4\delta_{Ln-L}, \quad \text{for } k = 2,\dots,16$$

where $r(m_K(s,t)) = |m_K(s,t)|$ is the distance from the point $m_K(s,t)$ to $(0,0)$.

We would like to examine one term in $H_{f,K}$, which is associated with s^3 and will show the general behavior of $H_{f,K}$.

$$\begin{aligned}\frac{\partial^3}{\partial s^3}f(m_K(\zeta,\eta)) &= \frac{\partial^3}{\partial s^3}(r^\alpha(m_K(s,t))\cdot g_1(m_K(s,t)))\Big|_{\substack{s=\zeta\\t=\eta}} \\ &= g_1(m_K(\zeta,\eta))\frac{\partial^3}{\partial s^3}(r^\alpha(m_K(s,t)))\Big|_{\substack{s=\zeta\\t=\eta}}\end{aligned} \tag{4.6}$$

$$+3\left\{\frac{\partial}{\partial s}g_1(m_K(s,t))\frac{\partial^2}{\partial s^2}(r^\alpha(m_K(s,t)))\right\}\Bigg|_{\substack{s=\zeta\\t=\eta}} \tag{4.7}$$

$$+3\left\{\frac{\partial^2}{\partial s^2}g_1(m_K(s,t))\frac{\partial}{\partial s}(r^\alpha(m_K(s,t)))\right\}\Bigg|_{\substack{s=\zeta\\t=\eta}} \tag{4.8}$$

$$+r^\alpha(m_K(\zeta,\eta))\frac{\partial^3}{\partial s^3}g_1(m_K(s,t))\big|_{\substack{s=\zeta\\t=\eta}}. \tag{4.9}$$

The magnitudes of (4.6)–(4.9) vary from $O(\delta^\alpha_{Ln-L})$ to $O(\delta^{\alpha+3}_{Ln-L})$, and $O(\delta^\alpha_{Ln-L})$ is the dominant term in (4.6)–(4.9). It follows that the coefficient of s^3 in $H_{f,K}$ is of $O(\delta^\alpha_{Ln-L})$. Consequently,

$$|H_{f,K}(s,t;\zeta,\eta)| \le O(\delta^\alpha_{Ln-L})$$

for any $(s,t)\in\sigma$.

Therefore,

$$\begin{aligned}
&\left|\int_{\Delta_K} f(q)\,dS - \sum_{j=4}^{6}\omega_{j,K}f(v_{j,K})\right|\\
&\quad\le O(\delta^2_{Ln-L})\int_\sigma\left|f(m_K(s,t)) - \sum_{j=1}^{6} f(m_K(\rho_j))l_j(s,t)\right|\,dS\\
&\quad\le O(\delta^2_{Ln-L})\int_\sigma |H_{f,K}(s,t;\zeta,\eta)|\,dS\\
&\quad\le O(\delta^{\alpha+2}_{Ln-L}).
\end{aligned}$$

Hence, (4.4) holds. □

LEMMA 2. *Under the assumptions in Lemma* 1, *let* $\beta=\alpha+2+\frac{\alpha-2}{L}$. *Then*

$$\left|\int_{B_l} f(q)\,dS - \sum_{\Delta_K\subset B_l}\sum_{j=4}^{6}\omega_{j,K}f(v_{j,K})\right| \le \begin{cases} O(2^{-4n-l\beta}) & \text{if } \alpha<3,\\ O(2^{-4n-(l/L)-5l}) & \text{if } \alpha\ge 3\end{cases} \tag{4.10}$$

for $l=1,\ldots,Ln-L-1$.

Proof. There are two types of triangles in B_l. Those triangles that are part of symmetric pairs of triangles (cf. Fig. 3) are of the first type and the remaining triangles are of the second type. By analysis of the $La+u$ refinement in §2, the number of triangles of the first type is $O(N_l)$, where N_l is the number of triangles in B_l. Then N_l is proportional to 4^{n-i}, where l is decomposed as $l=iL+i_1$ for $0\le i_1\le L-1$.

Since f is smooth on B_l, then by Taylor's error formula we have

$$f(m_K(s,t)) - \sum_{j=1}^{6} f(m_K(\rho_j))l_j(s,t) = H_{f,K}(s,t) + G_{f,K}(s,t;\zeta,\eta) \tag{4.11}$$

where

$$H_{f,K}(s,t) = \frac{1}{3!}\left[\left(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}\right)^3 f(m_K(0,0)) \right.$$
$$\left. - \sum_{j=1}^{6}\left(s_j\frac{\partial}{\partial s} + t_j\frac{\partial}{\partial t}\right)^3 f(m_K(0,0))l_j(s,t)\right],$$

$$G_{f,K}(s,t;\zeta,\eta) = \frac{1}{4!}\left[\left(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}\right)^4 f(m_K(\zeta,\eta)) \right.$$
$$\left. - \sum_{j=1}^{6}\left(s_j\frac{\partial}{\partial s} + t_j\frac{\partial}{\partial t}\right)^4 f(m_K(\zeta_j,\eta_j))l_j(s,t)\right].$$

We examine (4.11) and we can find the following.

First, $H_{f,K}(s,t)$ is a polynomial of degree 3. Second, the coefficients of $H_{f,K}(s,t)$ are combinations of $(v_{2,K} - v_{1,K})$ and $(v_{3,K} - v_{1,K})$. For instance, the coefficient of s^3 is

(4.12)

$$\frac{1}{3!}\frac{\partial^3}{\partial s^3}f(m_K(0,0))$$
$$= \frac{1}{3!}\left[\frac{\partial^3}{\partial x^3}f(m_K(\rho_1))(v_{3,x} - v_{1,x})^3 + 3\frac{\partial^3}{\partial x^2\partial y}f(m_K(\rho_1))(v_{3,x} - v_{1,x})^2(v_{3,y} - v_{1,y}) \right.$$
$$\left. + \frac{\partial^3}{\partial x\partial^2 y}f(m_K(\rho_1))(v_{3,x} - v_{1,x})(v_{3,y} - v_{1,y})^2 + \frac{\partial^3}{\partial y^3}f(m_K(\rho_1))(v_{3,y} - v_{1,y})^3\right],$$

where $v_{j,K} = (v_{j,x}, v_{j,y})$ for $j = 1,2,3$. For every symmetric pair of triangles, say Δ_1 and Δ_2, in B_l (see Fig. 3), let

$$m_1(s,t) = (v_3 - v_1)s + (v_2 - v_1)t + v_1,$$
$$m_2(s,t) = (v_5 - v_1)s + (v_4 - v_1)t + v_1,$$
$$v_j = (v_{j,x}, v_{j,y}).$$

Then

$$v_1 - v_2 = -(v_1 - v_4),$$
$$v_1 - v_3 = -(v_1 - v_5).$$

We now have for $H_{f,1}$ and $H_{f,2}$ that the coefficient of s^3 in $H_{f,1}$ is

(4.13)

$$\frac{1}{3!}\left[\frac{\partial^3}{\partial x^3}f(m_1(\rho_1))(v_{3,x} - v_{1,x})^3 + 3\frac{\partial^3}{\partial x^2\partial y}f(m_1(\rho_1))(v_{3,x} - v_{1,x})^2(v_{3,y} - v_{1,y}) \right.$$
$$\left. + 3\,\frac{\partial^3}{\partial x\partial^2 y}f(m_1(\rho_1))(v_{3,x} - v_{1,x})(v_{3,y} - v_{1,y})^2 + \frac{\partial^3}{\partial y^3}f(m_1(\rho_1))(v_{3,y} - v_{1,y})^3\right],$$

and the coefficient of s^3 in $H_{f,2}$ is

(4.14)

$$\frac{1}{3!}\left[\frac{\partial^3}{\partial x^3}f(m_2(\rho_1))(v_{5,x} - v_{1,x})^3 + 3\frac{\partial^3}{\partial x^2\partial y}f(m_2(\rho_1))(v_{5,x} - v_{1,x})^2(v_{5,y} - v_{1,y}) \right.$$
$$\left. + 3\,\frac{\partial^3}{\partial x\partial^2 y}f(m_2(\rho_1))(v_{5,x} - v_{1,x})(v_{5,y} - v_{1,y})^2 + \frac{\partial^3}{\partial y^3}f(m_2(\rho_1))(v_{5,y} - v_{1,y})^3\right].$$

Adding (4.13) and (4.14) gives us zero, and an analogous argument holds for the remaining coefficients. This means that cancellation happens on any symmetric pair of triangles. It follows that

$$\left| \int_{\Delta_1 \cup \Delta_2} f(q)\, dS - \sum_{K=1}^{2} \sum_{j=4}^{6} \omega_{j,K} f(v_{j,K}) \right| \leq O(\delta_l^2) \sum_{K=1}^{2} \int_{\sigma} |G_{f,K}(s,t;\zeta,\eta)|\, ds\, dt$$

where δ_l is the diameter of Δ_1 and Δ_2. This argument is based on Chien [6].

We now bound $G_{f,K}(s,t;\zeta,\eta)$ on $\Delta_K \subset B_l$ for $K = 1,2$. Since $1 \geq r(m_K(s,t)) \geq r_l > 0$ for $(s,t) \in \sigma$, and $r_l = O(2^{-(L+1)i})$, we have for $\alpha < 4$,

$$\begin{aligned} |G_{f,K}(s,t;\zeta,\eta)| &\leq O(\delta_l^4) \left\{ O(r(m_K(\zeta,\eta))^{\alpha-4}) + \sum_{j=1}^{6} O(r(m_K(\zeta_j,\eta_j))^{\alpha-4}) \right\} \\ &\leq O(\delta_l^4) O(r_l^{\alpha-4}) \\ &\leq O(\delta_l^4) O(2^{(4-\alpha)(L+1)i}). \end{aligned}$$

The first inequality arises from bounding the individual terms of $(\partial^4 f)/(\partial^a s \partial^b t), a + b = 4$, based on the kind of expansion done in (4.12). Otherwise, for $\alpha \geq 4$,

$$|G_{f,K}(s,t;\zeta,\eta)| \leq O(\delta_l^4).$$

So,

$$\left| \int_{\Delta_1 \cup \Delta_2} f(q)\, dS - \sum_{j=4}^{6} \omega_{j,K} f(v_{j,K}) \right| \leq \begin{cases} O(\delta_l^6) O(2^{(4-\alpha)(L+1)i}) & \text{if } \alpha < 4, \\ O(\delta_l^6) & \text{otherwise.} \end{cases}$$

The error contributed by triangles of the first type is as follows. If $\alpha < 4$,

$$\begin{aligned} & \left| \sum_{\Delta_K \text{ of first type}} \left(\int_{\Delta_K} f(q)\, dS - \sum_{j=4}^{6} \omega_{j,K} f(v_{j,K}) \right) \right| \\ & \leq O(N_l) O(\delta_l^6) O(2^{(4-\alpha)(L+1)i}) \\ & \leq O\left(2^{2n-2i} \frac{1}{2^{6n+6l}} \right) O(2^{(4-\alpha)(L+1)i}) \\ & \leq O(2^{-4n-l\beta}). \end{aligned} \tag{4.15}$$

For $\alpha \geq 4$,

$$\begin{aligned} & \left| \sum_{\Delta_K \text{ of first type}} \left(\int_{\Delta_K} f(q)\, dS - \sum_{j=4}^{6} \omega_{j,K} f(v_{j,K}) \right) \right| \\ & \leq O(N_l) O(\delta_l^6) \\ & \leq O(2^{-4n-(2l/L)-6l}). \end{aligned} \tag{4.16}$$

For the remaining triangles in B_l, which are those of the second type, the error contributed by a triangle Δ_K is $H_{f,K}(s,t;\zeta,\eta)$ by (4.5). By the fact that $r_l \leq$

$r(m_K(s,t)) \leq 1$ for $\Delta_K \subset B_l$, and the fact that their numbers is $O(\sqrt{N_l})$, then the error is bounded by

$$|H_{f,K}(s,t;\zeta,\eta)| \leq \begin{cases} O(\delta_l^3)O(r_l^{\alpha-3}) & \text{if } \alpha < 3, \\ O(\delta_l^3) & \text{otherwise,} \end{cases}$$

and

$$\left| \sum_{\Delta_K \text{ of second type}} \left(\int_{\Delta_K} f(q)\, dS - \sum_{j=4}^{6} \omega_{j,K} f(v_{j,K}) \right) \right| \leq \begin{cases} O(2^{-4n-l\beta}) & \text{if } \alpha < 3, \\ O(2^{-4n-(1/L)-5l}) & \text{otherwise.} \end{cases} \tag{4.17}$$

The total error over B_l is given by (4.10). □

LEMMA 3. *Under the assumptions in Lemma* 1,

$$\left| \int_{B_0} f(q)\, dS - \sum_{\Delta_K \subset B_0} \sum_{j=4}^{6} \omega_{j,K} f(v_{j,K}) \right| \leq O\left(\frac{1}{N^2}\right). \tag{4.18}$$

Proof. The function f is smooth on B_0, and B_0 is uniformly divided by triangular elements. By the results in [6], (4.18) follows. □

Combining the above lemmas, we get the following result, which gives the total error of integrating over S.

THEOREM 1. *Let f be of the form* (1.1). *Then*

$$\left| \int_S f(q)\, dS - \sum_{K=1}^{N} \sum_{j=4}^{6} \omega_{j,K} f(v_{j,K}) \right| \leq \begin{cases} O\left(\frac{\ln N}{N^2}\right) & \text{if } p_1 = 2, \\ O\left(\frac{1}{N^p}\right) & \text{otherwise}, \end{cases}$$

where N is the total number of triangles in the triangulation $p_1 = \frac{(\alpha+2)(L+1)}{2}$ and $p = \min\{p_1, 2\}$.

Proof. We first add all errors contributed by each $\Delta_K \subset B_1 \cup \cdots \cup B_{Ln-L-1}$. For $\alpha < 3$, if $p_1 \neq 2$ (i.e., $\beta \neq 0$),

$$\begin{aligned} \sum_{l=1}^{Ln-L-1} \left| \int_{B_l} f(q)\, dS - \sum_{\Delta_K \subset B_l} \sum_{j=4}^{6} \omega_{j,K} f(v_{j,K}) \right| &\leq \sum_{l=1}^{Ln-L-1} O(2^{-4n-l\beta}) \\ &= O\left(2^{-4n} \frac{2^{-\beta} - 2^{-\beta(Ln-L)}}{1 - 2^{-\beta}}\right) \\ &\leq O(2^{-2pn}) \\ &= O\left(\frac{1}{N^p}\right), \end{aligned}$$

while

$$\sum_{l=1}^{Ln-L-1} \left| \int_{B_l} f(q)\, dS - \sum_{\Delta_K \subset B_l} \sum_{j=4}^{6} \omega_{j,K} f(v_{j,K}) \right| \leq O\left(\frac{n}{2^{2n}}\right) \leq O\left(\frac{\ln N}{N^2}\right) \quad \text{for } \beta = 0.$$

When $\alpha \geq 3, \beta$ is nonzero. By a similar argument,

$$\sum_{l=1}^{Ln-L-1} \left| \int_{B_l} f(q)\, dS - \sum_{\Delta_K \subset B_l} \sum_{j=4}^{6} \omega_{j,K} f(v_{j,K}) \right| \leq O\left(\frac{1}{N^p}\right) \quad \text{for } \alpha \geq 3.$$

Combining the above with Lemmas 1 and 3, we have

$$\begin{aligned}
&\left| \int_S f(q)\, dS - \sum_{K=1}^{N} \sum_{j=4}^{6} \omega_{j,K} f(v_{j,K}) \right| \\
&\leq \left| \int_{B_{Ln-L}} f(q)\, dS - \sum_{\Delta_K \subset B_{Ln-L}} \sum_{j=4}^{6} \omega_{j,K} f(v_{j,K}) \right| \\
&\quad + \sum_{l=1}^{Ln-L-1} \left| \int_{B_l} f(q)\, dS - \sum_{\Delta_K \subset B_l} \sum_{j=4}^{6} \omega_{j,K} f(v_{j,K}) \right| \\
&\quad + \left| \int_{B_0} f(q)\, dS - \sum_{\Delta_K \subset B_0} \sum_{j=4}^{6} \omega_{j,K} f(v_{j,K}) \right| \\
&\leq O\left(\frac{1}{N^{p_1}}\right) + O\left(\frac{1}{N^p}\right) + O\left(\frac{1}{N^2}\right) \\
&\leq O\left(\frac{1}{N^p}\right) \quad \text{for } \beta \neq 0.
\end{aligned}$$

And it is $O(\ln N/N^2)$ for $\beta = 0$. The quantity p_1 is defined in Lemma 1, and $p = \min\{p_1, 2\}$. □

COROLLARY. *Let L be any positive integer greater than $\frac{4}{\alpha+2} - 1$. Then the error for evaluating the integral over S is $O(1/N^2)$. In particular, a $1a + u$ refinement gives an error of $O(1/N^2)$ for any $\alpha > 0$.*

THEOREM 2. *Let f be of the form* (1.2). *Then*

$$\left| \int_S f(q)\, dS - \sum_{K=1}^{N} \sum_{j=4}^{6} \omega_{j,K} f(v_{j,K}) \right| \leq O\left(\frac{\ln N}{N^{p_1}}\right) + O\left(\frac{1}{N^2}\right)$$

where N is the total number of triangles in the triangulation and $p_1 = \frac{(\alpha+2)(L+1)}{2}$.

Proof. The proof is analogous to that of Theorem 1. □

COROLLARY. *Let L be any positive integer greater than $\frac{4}{\alpha+2} - 1$. Then the error for evaluating the integral over S is $O(1/N^2)$. For $\alpha > 0$, a $1a + u$ refinement still gives an error of $O(1/N^2)$.*

5. Numerical examples. We give numerical examples using the method analyzed in §4. The method was implemented with a package of programs written by Atkinson, which is described in [1] and [4]. All examples were computed on a Hewlett Packard workstation in double precision arithmetic.

Example 1. Let $S = \left\{(x, y) \in \mathbf{R}^2 | 0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}\right\}$,

$$f(x, y) = r^\alpha, \qquad \alpha > -2. \tag{5.1}$$

TABLE 1
Evaluation with $L = 1$.

		$\alpha = .1$		$\alpha = .5$	
n	N	Error	Order	Error	Order
1	4	−2.36E-4		−9.47E-4	
2	28	−9.00E-6	1.68	2.90E-6	3.10
3	124	3.97E-7	2.10	3.22E-6	−.23
4	508	7.13E-8	1.22	2.91E-7	1.71
5	2044	6.95E-9	1.67	2.10E-8	1.89
6	8188	5.70E-10	1.81	1.40E-9	1.95

TABLE 2
Evaluation with $\alpha = -1$.

		$L = 1$			$L = 3$	
n	N	Error	Order	N	Error	Order
1	4	1.49E-1		4	1.49E-1	
2	28	4.01E-2	0.68	52	9.51E-3	1.08
3	124	1.07E-2	0.93	244	5.61E-4	1.83
4	508	2.53E-3	0.98	1012	3.35E-5	1.98
5	2044	6.32E-4	1.00	4084	2.00E-6	2.02
6	8188	1.58E-4	1.00			

The results are given in Table 1 for $\alpha = 0.1, 0.5$ with $L = 1$. The column labeled *Order* gives the value

$$p_n = \frac{\ln |E_n/E_{n+1}|}{\ln(N_{n+1}/N_n)}$$

where E_n is the error at level n. Since the theoretical result shows that the error is $O(1/N^2)$, we expect that p_n will converge to 2.

Table 2 gives the results for $\alpha = -1$ with $L = 1$ and $L = 3$. The empirical orders of convergence p_n approach 1 and 2, respectively, as expected.

6. Generalization. We have presented results for only the planar wedge, while using polynomial interpolation of degree 2 to approximate the integrand and the integration region. Any other degree of interpolation could also have been used. In such cases, the definition of the nodes will change appropriately, but the definition of the triangulation will remain the same, and we will use the $La + u$ refinement.

In addition to using quadratic interpolation, we have also examined the use of linear, cubic, and quartic interpolation. For linear and cubic interpolation, the function value at the origin is needed in the numerical integration. We simply let $f(0,0) = 0$. The results are consistent with the kind of results we have obtained for the quadratic case, and they are as follows.

Suppose that we use interpolation of degree d to approximate both the integrand and the wedge S. Let $\{q_1, \ldots, q_v\}$ be the node points in the unit simplex and let $\{l_1, \ldots, l_v\}$ be the basis functions in the Lagrange form, where $v = \frac{(d+1)(d+2)}{2}$. The points $\{q_1, \ldots q_v\}$ will be equally spaced over σ and of the form $\left(\frac{i}{d}, \frac{j}{d}\right), 0 \le i + j \le d$. Define the interpolating operation

$$\mathcal{P}_N h(s,t) = \sum_{j=1}^{v} h(q_j) l_j(s,t).$$

The numerical integration we use is based on integrating this interpolation polynomial; i.e.,

$$\int_\sigma h(s,t)\,ds\,dt \approx \int_\sigma \mathcal{P}_N h(s,t)\,ds\,dt = \sum_{j=1}^{v} \omega_j^{(d)} h(q_j).$$

Assume that the integrand is of the form (1.1). Then with the $La+u$ refinement, we have

$$\left| \int_S f(q)\,dS - \sum_{K=1}^{N_n} \sum_{j=1}^{v} \omega_{j,K}^{(d)} f(v_{j,K}) \right| \le \begin{cases} O\left(\frac{\ln Nn}{N_n^{d^*}}\right) & \text{if } p_1 = d^*, \\ O\left(\frac{1}{N_n^p}\right) & \text{otherwise,} \end{cases}$$

where N_n is the number of triangular elements in the triangulation. The order $p = \min\{p_1, d^*\}, p_1 = \frac{(\alpha+2)(L+1)}{2}, d^* = \frac{d+2}{2}$ when d is an even number, and $d^* = \frac{d+1}{2}$ when d is an odd number. The proof is completely analogous to that given earlier for the quadratic case. In addition, we need the results for smooth integrands as stated in [6].

The results can be generalized to other integration regions S, for example, a wedge with the central angle larger than π, triangles, squares, regions containing the origin as an interior point, etc. We also can generalize this to curved surfaces in $\mathbf{R}^3$, with the integrand having a point singularity of the type given in (1.1). We omit statements of these results because they are straightforward.

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