Rates of Convergence
Iteration Methods for Poisson's Equation

Consider the discretization (8.8.1) of page 557 for Poisson's equation. Define

$$\xi = 1 - 2\sin^2\left(\frac{\pi}{2N}\right), \quad \omega^* = \frac{2}{1+\sqrt{1-\xi^2}}$$

Rate of Convergence	
Jacobi	ξ
Gauss-Seidel	ξ^2
SOR	$\omega^* - 1$

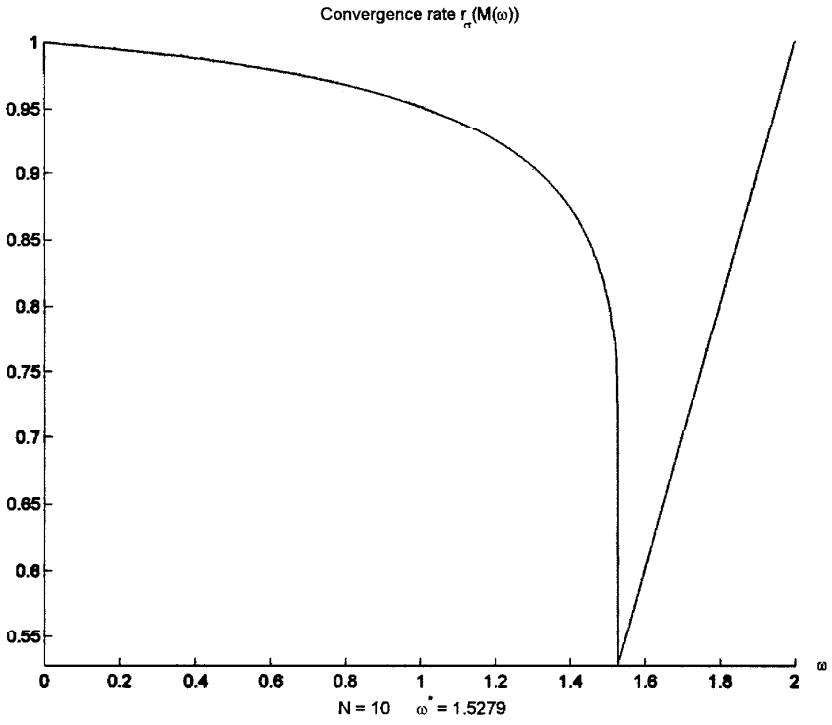
For the SOR method,

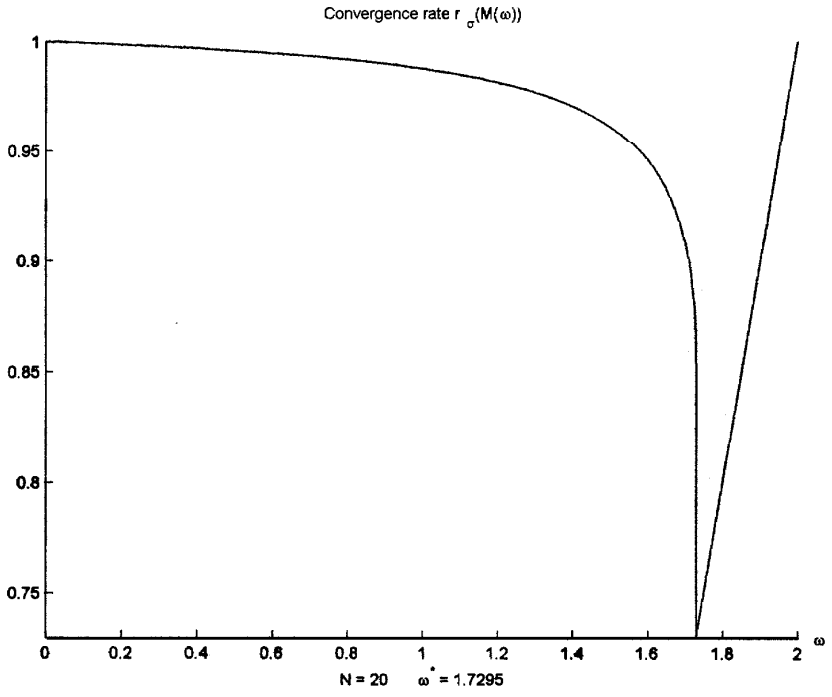
$$r_\sigma(M(\omega)) = \begin{cases} \frac{1}{2}\left[\omega\xi + \sqrt{(\omega\xi)^2 + 4(1-\omega)}\right], & 0 \le \omega \le \omega^* \\ \omega - 1, & \omega^* \le \omega \le 2 \end{cases}$$

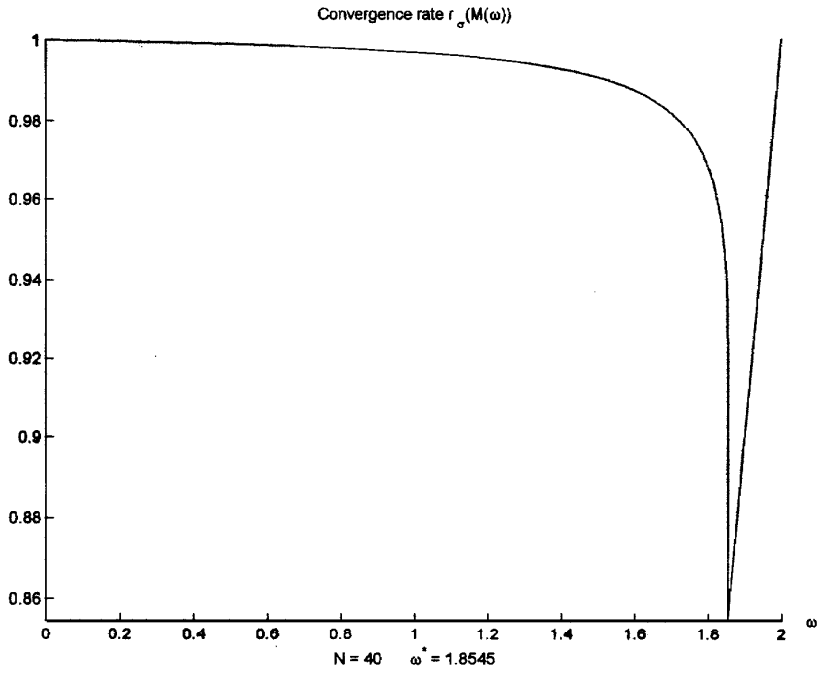
Let M_J, M_{GS}, and M_{SOR} denote the number of iterations needed to reduce the iteration error by a factor of 10, for the Jacobi, Gauss-Seidel, and SOR methods, respectively.

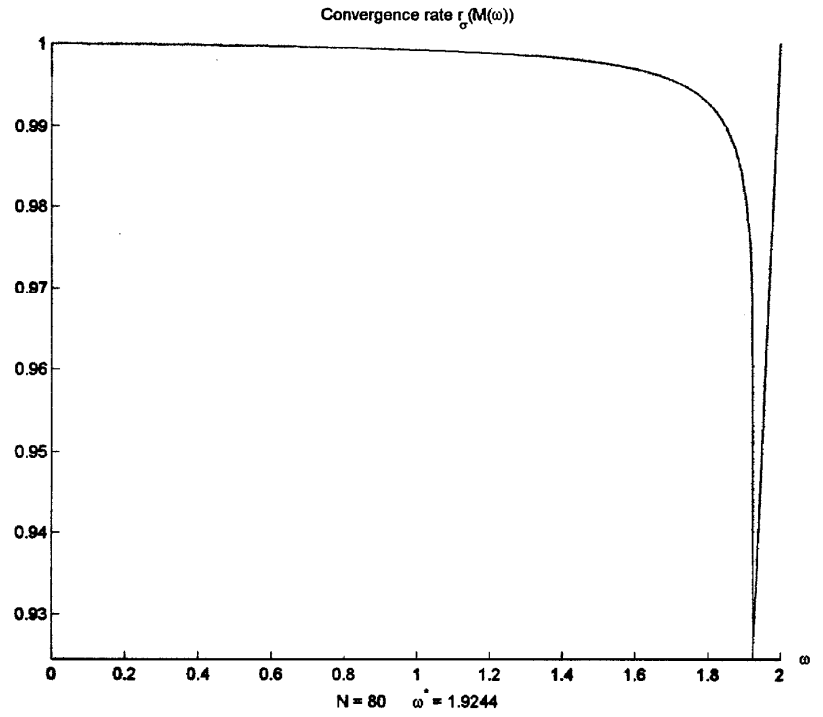
	$N = 10$	$N = 20$	$N = 40$	$N = 80$
ξ	.9510565	.9876883	.9969173	.99922904
ξ^2	.9045085	.9755283	.9938442	.99845867
$\omega^* - 1$	.527864	.729453	.854497	.924446
M_J	46	186	746	2985
M_{GS}	23	93	373	1493
M_{SOR}	4	8	15	31

On the following graphs, note that the vertical scales are different with each graph.


Convergence rate $r_\sigma(M(\omega))$
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
ω
N = 10 $\omega^* = 1.5279$


Convergence rate $r_\sigma(M(\omega))$
1
0.95
0.9
0.85
0.8
0.75
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
ω
N = 20 $\omega^* = 1.7295$


Convergence rate $r_\sigma(M(\omega))$
1
0.98
0.96
0.94
0.92
0.9
0.88
0.86
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
ω
N = 40 $\omega^* = 1.8545$


Convergence rate $r_\sigma(M(\omega))$
1
0.99
0.98
0.97
0.96
0.95
0.94
0.93
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
ω
N = 80 $\omega^* = 1.9244$