Systems of linear equations

A system of 2 linear equations in 2 unknowns looks like

$$ax + by = u,$$
$$cx + dy = v,$$

where a, b, c, d, u, v are constants (and $x.y$ are the unknowns).

In writing larger systems, we quickly run out of letters to use for the names of the various constant coefficients (like a, b, c, d above). Thus we go to an indexed notation, which will turn out to pay added dividends not too far down the road, when we develop some matrix methods for solving linear systems. For example, in an indexed (or subscripted) notation, we might write a 2×2 system as above as

$$a_{11}x_1 + a_{12}x_2 = b_1,$$
$$a_{21}x_1 + a_{22}x_2 = b_2.$$

Note that we economize on letters, but at the cost of having little subscripts dancing around in all our formulas. But on the good side, the subscripts accomplish a nice accounting chore for us. For example, the first index on the a tells us which equation this coefficient comes from, and the second index tells us which unknown the coefficient is attached to (i.e. which unknown it multiplies).

There's some (traditional) abuse of notation in this – we should really say, for example, $a_{1,2}$. But we omit the tiny comma unless some of the indices take multi-digit values (10 and beyond).

Given this, it's not too hard to see what a 3×3 system should look like:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3. \end{aligned}$$

The above are all square systems of equations – that is, they have the same number of equations as unknowns. But there's no reason that a linear system has to be square, and many systems occurring "in real life" are not square. For example,

$$\begin{aligned} 5x - 3y + 7z &= 11, \\ \tfrac{1}{2}x - y + 40z &= -13 \end{aligned}$$

is a 2×3 system, meaning it is a system of 2 equations in 3 unknowns. The system

$$\begin{aligned} x + y &= 1, \\ x - y &= 2, \\ 5x + y &= 5 \end{aligned}$$

is a system of 3 equations in 2 unknowns; that is, a 3×2 system.

Problem: *Write down a template for what a* 5×4 *system should look like.*

Since such a system will have $5 \cdot 4 = 20$ coefficients on its left-hand side, we use an indexed notation:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= b_1, \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= b_2, \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= b_3, \\
a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 &= b_4, \\
a_{51}x_1 + a_{52}x_2 + a_{53}x_3 + a_{54}x_4 &= b_5.
\end{aligned}$$

When we get a fairly large system, the idea of using nothing but substitution to solve becomes unwieldy – one just has to attempt anything larger than a 3×3 system to see this. Even a 3×3 system becomes pretty messy using substitution. One method that works well (really the most computable, for several reasons) is <u>Gaussian elimination</u>, sometimes called simply <u>elimination</u>.

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Example: To illustrate the method, consider the system

$$\begin{aligned} 4x + 3y + 2z &= 0, \\ x + y + 2z &= 0, \\ 2x - 2y + 2z &= 11. \end{aligned}$$

Interchanging the first two equations, $\boxed{1} \leftrightarrow \boxed{2}$, we have

$$\begin{aligned} x + y + 2z &= 0, \\ 4x + 3y + 2z &= 0, \\ 2x - 2y + 2z &= 11. \end{aligned}$$

The strategy now is to add multiples of the first equation to the second and third, eliminating the x-term in each. To describe this, one sometimes says that the x-term in the first equation is being used as a <u>pivot</u>. Implementing

$$\begin{aligned} \boxed{2} &\to \boxed{2} - 4 \cdot \boxed{1}, \\ \boxed{3} &\to \boxed{3} - 2 \cdot \boxed{1}, \end{aligned}$$

we have

$$\begin{aligned} x + y + 2z &= 0, \\ -y - 6z &= 0, \\ -4y - 2z &= 11. \end{aligned}$$

Multiplying the second equation by -1, i.e. $\boxed{2} \to -\boxed{2}$, we get

$$\begin{aligned} x + y + 2z &= 0, \\ y + 6z &= 0, \\ -4y - 2z &= 11. \end{aligned}$$

We now use the y-term in the second equation as a new pivot, to knock out the y-term below it: implementing

$$\boxed{3} \to \boxed{3} + 4 \cdot \boxed{2},$$

we get

$$\begin{aligned} x + y + 2z &= 0, \\ y + 6z &= 0, \\ 22z &= 11. \end{aligned}$$

At this point, we could also knock out the y-term in the first equation. But an important strategic point is that it's usually better to wait on this, as the work we would be doing might be wasted. (The coefficient we are zeroing out may be bounced around some before we are done, and we may just have to zero it out again later.)

Dividing the last equation by 22, we have

$$\begin{aligned} x + y + 2z &= 0, \\ y + 6z &= 0, \\ z &= \tfrac{1}{2}. \end{aligned}$$

Note that this last equation says what the value of z has to be for any solution (x, y, z) (though we are not yet sure there **are** any solutions).

We continue with a process which is sometimes called <u>back elimination</u>. Roughly speaking, we now move from right to left (instead of left to right) and eliminate terms. To zero out the z-terms in the first and second equations, we implement

$$\boxed{1} \longrightarrow \boxed{1} - 2 \cdot \boxed{3},$$
$$\boxed{2} \longrightarrow \boxed{2} - 6 \cdot \boxed{3}$$

to get

$$\begin{aligned} x + y \quad &= -1, \\ y \quad &= -3, \\ z &= \tfrac{1}{2}. \end{aligned}$$

Now implementing

$$\boxed{1} \longrightarrow \boxed{1} - \boxed{2},$$

we have

$$\begin{aligned} x \quad\quad &= 2, \\ y \quad &= -3, \\ z &= \tfrac{1}{2}. \end{aligned}$$

We have found that the unique solution of the original system is

$$(x, y, z) = (2, -3, \tfrac{1}{2}).$$

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Looking back at the last example, and not thinking about **strategy** for the moment, we really had only 3 **tactics**, or elementary equation operations:

1. interchange 2 equations;
2. multiply an equation by a nonzero constant;
3. add a constant multiple of one equation to another equation.

This is, in fact, a complete list of the tactics needed for all elimination computations. In the second tactic, it's clear that we have to include the word "nonzero." For clearly, if we multiply an equation by 0, we end up with the equation $0 = 0$, thus wiping out any information that the equation might have originally carried.

It is natural to worry whether implementation of any of these tactics, or of a series of them, might somehow lose information, or inject new information that was not in the original system. The good news is that there is nothing to worry about on this score. This is due to the fact that:

> **Each elementary equation operation may be reversed by an operation of the same type.**

Indeed, an interchange operation $\boxed{i} \leftrightarrow \boxed{j}$ is reversed by the same operation. The operation $\boxed{\text{i}} \to c \cdot \boxed{\text{i}}$, where $c \neq 0$, is reversed by

$$\boxed{\text{i}} \to \frac{1}{c} \cdot \boxed{\text{i}}.$$

(Note that c had to be nonzero in order to form $1/c$.) Finally, the operation

$$\boxed{\text{i}} \to \boxed{\text{i}} + k \cdot \boxed{\text{j}},$$

where k is a constant, is reversed by

$$\boxed{\text{i}} \to \boxed{\text{i}} - k \cdot \boxed{\text{j}}.$$

(There is no need to assume k is nonzero here, though doing either of these operations with $k = 0$ has no effect on any equation.)

The reversibility of all our elementary equation operations guarantees that we never lose nor gain information. For any solution of a system is still a solution after applying an elementary operation. This means that the solution set of the new system contains that of the old system. But applying the reverse operation, we exchange the roles of “old” and “new.” Thus the solution sets of the two systems both contain and are contained in each other; that is, they are the same.

A 3×3 system need not have a unique solution, as the following two examples show. Qualitatively, two other things can happen – no solutions, in which case the system is said to be inconsistent, and continuous families of solutions (in particular, infinitely many solutions).

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Example: Consider the system

$$\begin{aligned} x + 4y + 5z &= 3, \\ 7x - 3y + 2z &= -4, \\ 9x + 5y + 12z &= 3. \end{aligned}$$

Implementing

$$\boxed{2} \rightarrow \boxed{2} - 7 \cdot \boxed{1},$$
$$\boxed{3} \rightarrow \boxed{3} - 9 \cdot \boxed{1}.$$

we get

$$x + 4y + 5z = 3,$$
$$-31y - 33z = -25,$$
$$-31y - 33z = -24.$$

If we're alert, we can see immediately that this is inconsistent – the last two equations set the same quantity equal to both -25 and -24. Even if we don't see this, we might decide to implement

$$\boxed{3} \rightarrow \boxed{3} - \boxed{2};$$

the third equation of the new system is then $0 = 1$, an obvious contradition. Thus the system above has no solutions.

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Example: Tweak the last example slightly to obtain the system

$$\begin{aligned} x + 4y + 5z &= 3, \\ 7x - 3y + 2z &= -4, \\ 9x + 5y + 12z &= 2. \end{aligned}$$

Implementing

$$\begin{aligned} \boxed{2} &\to \boxed{2} - 7 \cdot \boxed{1}, \\ \boxed{3} &\to \boxed{3} - 9 \cdot \boxed{1}. \end{aligned}$$

we get

$$\begin{aligned} x + 4y + 5z &= 3, \\ -31y - 33z &= -25, \\ -31y - 33z &= -25. \end{aligned}$$

This time the last two equations don't contradict each other. If we implement

$$\boxed{3} \to \boxed{3} - \boxed{2},$$

the third equation is just $0 = 0$ a <u>tautology</u> (i.e. a statement that's always true). Thus our original system is equivalent to the 2×3 system

$$x + 4y + 5z = 3,$$
$$-31y - 33z = -25.$$

Implementing

$$\boxed{2} \rightarrow -\frac{1}{31} \cdot \boxed{2},$$

we get

$$x + 4y + 5z = 3,$$
$$y + \tfrac{33}{31}z = \tfrac{25}{31}.$$

Implementing

$$\boxed{1} \rightarrow \boxed{1} - 4 \cdot \boxed{2},$$

we get

$$x + \tfrac{23}{31}z = -\tfrac{7}{31},$$
$$y + \tfrac{33}{31}z = \tfrac{25}{31}.$$

And this is as far as we can go with elimination – this last form of the system is a description of the solution set:

$$z \text{ can be anything;}$$
$$x = -\tfrac{23}{31}z - \tfrac{7}{31};$$
$$y = -\tfrac{33}{31}z + \tfrac{25}{31}.$$

There are also descriptions of this same solution set in which either x or y (rather than z) plays the role of the free variable; z was chosen here just because it was written last in each equation of our system.

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