Exact constants in dilation theory

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The Caratheodory-Fejer problem

Given $a_0, a_1, \ldots, a_n \in \mathbb{C}$, consider all lower-triangular infinite Toeplitz matrices of the form

$$\begin{pmatrix} a_0 & & & & & & & \\ a_1 & a_0 & & & & & & \\ a_2 & a_1 & a_0 & & & & & \\ \cdots & \cdots & \cdots & \cdots & & & & \\ a_n & \cdots & \cdots & \cdots & a_0 & & & \\ ? & a_n & \cdots & \cdots & \cdots & a_0 & & \\ ? & ? & a_n & \cdots & \cdots & \cdots & a_0 & \\ ? & ? & ? & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

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What is the smallest norm of such a matrix?

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A Toeplitz matrix is a matrix whose entries are constant along diagonals.

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

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The norm of this operator is the L^∞ norm of f.

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Problem

Given $a_0, a_1, \ldots, a_n$, what is the minimal norm of an element of H^∞ whose Fourier series begins

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots ?$$

A purely function-theoretic question.

The Caratheodory-Fejer problem

We are asking about dilations of the matrix

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...and hence norm at least $\|T_a\|$.

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Theorem

Given the tuple $a_0, \ldots, a_n$,

- *There is an element of* H^∞ *with Fourier series beginning* $a_0 + a_1 z + \cdots + a_n z^n$ *and norm* $\|T_a\|$.

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- *This function is unique, and is $\|T_a\|$ times a finite Blaschke product of order at most n.*

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In the 1960s, Sarason showed this result could be well understood abstractly.

Connects with work of Nagy, Foias, and others.

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- The set of $(n+1) \times (n+1)$ lower triangular Toeplitz matrices is the commutant of an operator.
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And now, the actual subject of the talk.

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Consider the *self-adjoint* matrix

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What is the minimal norm of an dilation of S_a to a Toeplitz operator on H^2?

Equivalently: how to "extend" the real-valued Laurent polynomial

$$\overline{a_n} z^{-n} + \cdots + \overline{a_1} z + a_0 + a_1 z + \cdots + a_n z^n$$

to an element of L^∞ of minimal norm?

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This is *not* the CF problem.

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This problem is different because its answer is different.
In general it is not possible to find a Toeplitz dilation of S_a of norm $\|S_a\|$.

An example

Consider $n = 1$, $a_0 = 0$, and $a_1 = 1$.

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Can we extend $\frac{1}{z} + z$ to an element $f \in L^\infty$ of norm 1?

An example

Any $f \in L^\infty$ satisfies

$$\|f\|_\infty \geq \|f\|_2.$$

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Any $f \in L^\infty$ satisfies

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- If we could, then $||f||_2 = \sqrt{2}$.
- Then $\hat{f}(k) = 0$ for all $|k| > 1$ and $f(z) = \frac{1}{z} + z$.
- But this f has norm 2.

An aside

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More about this later.

The problems

Given $a = (a_0, a_1, \ldots, a_n) \in \mathbb{C}^{n+1}$ with $a_0 \in \mathbb{R}$, define

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How large can $\frac{m_a}{\|S_a\|}$ get?

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One formulation of the second problem: consider the sequence

$$D_n = \sup_{a \in \mathbb{C}^{n+1}} \frac{m_a}{\|S_a\|}, \qquad n = 1, 2, \ldots$$

Here the supremum is runs over all nonzero tuples with $a_0 \in \mathbb{R}$.

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One can show without too much work that if $m \mid n$ then $D_m \leq D_n$.

That is about it.

Results in this area

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These authors were not trying to explicitly evaluate $\sup D_n$.
There is little reason to suspect that it actually is 2.

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The lower bound 1 can be improved; we will see that in fact

$$D_n \in \left[\frac{\pi}{2}, 2\right], \qquad n \geq 1.$$

Note $\frac{\pi}{2} \approx 1.570796$.

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I have a similar guess regarding D_5.

Understanding D_n

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And m_a the smallest norm of a Toeplitz dilation to H^2.

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- D_n is the operator norm of the inverse of $\tilde{P}_n$.

D_n measures the extent to which P_n fails to be a quotient mapping. (In the CF problem, the compression *is* a quotient mapping.)

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- Any real $f \in L^\infty$ satisfying $\hat{f}(k) = a_k$ for $-n \leq k \leq n$ is of the form

  $$2 \operatorname{Re} h$$

  for h an analytic function on $\mathbb{D}$ with Taylor series

  $$\frac{a_0}{2} + a_1 z + a_2 z^2 + \cdots a_n z^n + \cdots$$

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- Any real $f \in L^\infty$ satisfying $\hat{f}(k) = a_k$ for $-n \leq k \leq n$ is of the form

$$2 \operatorname{Re} h$$

  for h an analytic function on $\mathbb{D}$ with Taylor series

$$\frac{a_0}{2} + a_1 z + a_2 z^2 + \cdots a_n z^n + \cdots$$

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  I will let $a' = (\frac{a_0}{2}, a_1, \ldots, a_n)$ denote this altered tuple of Fourier coefficients.
- In general h will not be bounded; only in H^2.

Reformulation

Thus m_a is the solution to the problem

$$\text{minimize } 2\|\operatorname{Re} h\|_\infty$$

where h runs over the analytic functions beginning

$$a_0' + a_1' z + \cdots + a_n' z^n \cdots$$

Reformulation

Choose h attaining the minimum. Necessarily h maps $\mathbb{D}$ into the strip

$$S = \left\{ w \in \mathbb{C} : |\operatorname{Re} w| < \frac{m_a}{2} \right\}.$$

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This is a holomorphic bijection from the interior of S to $\mathbb{D}$.
It maps the right boundary line of S bijectively onto the "upper half" of $\mathbb{T}$ and the left boundary line of S bijectively onto the "lower half" of $\mathbb{T}$.

Calculating m_a

Since ϕ is analytic,

$$(\phi \circ h)(z) = b_0 + b_1 z + \cdots + b_n z^n + \cdots$$

where $b = (b_0, \ldots, b_n)$ depends only on a' (and not on the unknown coefficients of h.)

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We say $\phi \circ h$ "begins with" b.
If k is analytic and begins with b, then $\phi^{-1} \circ b$ is analytic and begins with a'.

Calculating m_a

By the CF problem, there is $k \in \mathbb{R}$ and a finite Blaschke product P of order at most n with the property that

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Since $\phi \circ h$ is one of these, and $\|\phi \circ h\|_\infty = 1$ by definition,

$$k \leq 1.$$

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This would give us an analytic $h' : \mathbb{D} \to \mathbb{C}$ beginning with a' whose real part was too small.

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Since P is at most an n-to-1 local homeomorphism of $\mathbb{T}$ onto itself, $2 \operatorname{Re} h$ has at most $2n$ discontinuities, and is equal to m_a on at most n subintervals of $\mathbb{T}$.

Conclusion

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Theorem (Sarason)

If F_n denotes the set of all $\{\pm 1\}$ valued functions on $\mathbb{T}$ that assume the value 1 on at most n disjoint subintervals of $\mathbb{T}$, then

$$D_n = \sup_{f \in F_n} \frac{1}{\|T_f\|}$$

where T_f denotes the Toeplitz operator on $[1, z, \cdots, z^{n+1}]$ induced by multiplication by f.

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- Partitions of $\mathbb{T}$ into n intervals and how the norm of $\|T_f\|$ depends on the partition.
- The lower bound $\frac{\pi}{2}$ for D_n corresponds to considering f taking the value 1 on n equally spaced intervals of equal length.
- I originally thought that $D_n = \frac{\pi}{2}$ for all n (why not?)
- My better estimates for D_n have come thinking not in terms of the partitions themselves but the corresponding Blaschke products which are sometimes easier to work with.

This is a work in progress; hopefully there will be more to say soon.