RESEARCH STATEMENT

DIEGO RIBEIRO MOREIRA

Department of Mathematics, University of Iowa,
14 Maclean Hall, 52242, USA
E-mail: dmoreira@math.uiowa.edu
Webpage: www.math.uiowa.edu/∼dmoreira

1. Overview

I start by providing a general overview of the topics in which my research is concerned. My main research interests are partial differential equations, geometric measure theory and harmonic analysis. My research has focused on the area of nonlinear elliptic partial differential equations and also in the field of harmonic analysis.

The research in nonlinear elliptic equations is one of the most developed in Mathematics and of great importance because its applications in broader scientific disciplines such as fluid dynamics, phase transitions and mathematical finance. The past few decades have witnessed many new developments in the theory of fully nonlinear equations and homogenization issues promoting a better understanding of problems in material science, pricing of american options and combustion theory just to mention few examples.

I am interested to investigate qualitative and geometric properties of solutions and their level sets. The topics that occupy significant importance in my research include the regularity theory, geometry and asymptotic behavior of solutions.

In particular, I have studied free boundary problems. They are the central subject in the study of phenomena where phase transitions occur and arise when one attempts to describe a discontinuous change of behavior in a physical or biological quantity. Applications appear in stopping time for optimal control, ground water hydrology, plasticity theory, optimal design and problems in superconductivity. Typical examples are the evolution of an ice-water mixture, an elastic membrane constrained to stay within a given region and the description of laminar flames as an asymptotic limit for high energy activation.

I have worked in the theory of free boundary problems of flame propagation type ([M1], [M2],[MT3]), geometric measure theory of free boundaries of solutions to fully nonlinear and quasi-linear elliptic equations ([MW1]) and also boundary behavior of solutions in domains with low regularity ([MW2], [MW3]).

Harmonic analysis has continuously showed itself as one of the cornerstones of the modern Mathematics. It is a fundamental support area for many others disciplines such as PDEs, analytic number theory, fluid dynamics and differential geometry. It has a wide range of applications in engineering and applied mathematics in areas such as image and signal processing, wavelets and approximation theory.

My interests in this field include the theory of maximal and singular integral operators. I have been investigating regularity properties of multi-linear maximal operators of Hardy-Littlewood type with respect to the weak differentiability. More specifically, how the action of multi-linear maximal operators alters the (weak) differentiability properties of functions. This worked was developed in [CM1], [CM2].

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In the field of nonlinear functional analysis, I studied the stability of the weak convergence under the action of some general class of nonlinearities called Nemytskii nonlinearities in Lebesgue and Sobolev spaces ([MT1],[MT2]). The motivation for this study comes from the fact that many differential and integral equations, which models many problems in mathematical physics, can be seen as nonlinear functional equations in infinite dimensional Banach spaces where the lack of local compactness in the norm topology poses difficulties for theory.

In the sequel, I present the results I have obtained and my plans for future research in the three areas: (2) nonlinear elliptic PDEs and free boundary problems, (3) harmonic analysis and (4) nonlinear functional analysis.

2. Nonlinear Elliptic Partial Differential Equations and Free Boundary Problems

2-A. Free boundary Problems of Flame propagation type

Regularizing Methods in Free Boundary Problems. Regularizing methods in free boundary problems (FBPs) are models for a wide spectrum of problems in nature. They are of particular interest in the theory of flame propagation to describe laminar flames as an asymptotic limit for high energy activation. These methods go back to Zeldovich and Frank-Kamenetski, [ZF], in 1938. However, the rigorous mathematical study was postponed until recently with the pionerring works of Berestycki-Caffarelli-Nirenberg [BCN] and Caffarelli-Vazquez [CV].

In the last 19 years, many authors have given attention to the study of the limit as $\varepsilon \to 0$ of solutions to the elliptic equation of the form

$$\Delta u = \beta_\varepsilon(u) \tag{2.1}$$

where β_ε is a regularization that concentrates in the $\varepsilon-$strip $\{0 < u < \varepsilon\}$ with a peak of height $1/\varepsilon$ and approximates the Dirac measure at the origin. Mathematically, β_ε is given by $\beta_\varepsilon(s) = 1/\varepsilon\beta(s/\varepsilon)$ and β is a Lipschitz continuous function, with $\beta > 0$ in $(0,1)$, $\text{supp}(\beta) = [0,1]$ and $\int \beta = M > 0$. We observe that, in general, the comparison principle is not available for equations of the form (2.1). This way, uniqueness for solutions u_ε are not expected to hold. It is known from a series of interesting papers of Luis Caffarelli, Claudia Lederman and Noemi Wolanski ([CLW1],[CLW2],[LW]) that under certain geometric assumptions about a limit function u_0 and its free boundary, this function becomes a viscosity solution of the following free boundary problem

$$\begin{cases} \Delta u &= 0 \quad \text{in } \Omega \setminus \{u > 0\} \\ (u_\nu^+)^2 - (u_\nu^-)^2 &= 2M \quad \text{on } F(u) := \Omega \cap \partial\{u > 0\}, \end{cases} \tag{2.2}$$

and the free boundary $F(u)$ is locally a $C^{1,\alpha}$ surface. These extra geometric assumptions are necessary if one intends to obtain further regularity results since there are limits u_0 for which the free boundary condition, $(u_\nu^+)^2 - (u_\nu^-)^2 = 2M$ is not satisfied in the classical sense in any portion of the free surface ([CLW1], remark 5.1).

Since multiplicity of solutions may occur, we can raise the following question: Can we actually produce special solutions u_ε enjoying some geometric properties that will allow us to study the geometry and regularity of the limit problem ? In the papers [M1],[M2] and [MT3] we dealt with this question in a more general way. Indeed, we constructed solutions u_ε to equations more general than (2.1), where the level surfaces enjoy some geometric properties which are uniform in ε. These properties could be transferred to the free boundary of a limit function. Then, the limit problem was studied. The construction of special solutions u_ε took place in the nonvariational ([M1],[M2]) and varitional ([MT3]) settings. We explain them separately.

Novariational Free Boundary Problems. In [CJK], Luis Caffarelli (Univ. Texas at Austin), David Jerison (MIT) and Carlos Kenig (Univ. of Chicago) proved some new monotonicity results so that it applies to inhomogeneous equations in which the right-hand side of the equation does not need to vanish on the free boundary. The new versions of the monotonicity theorem led to some existence and regularity results to the Prandtl-Batchelor equation, which plays an important role in fluid dynamics. In connection with these results, a uniform Lipschitz estimates for solutions of the family (2.3) of semilinear equations was proven.

$$\Delta u = \beta_\varepsilon(u)F(\nabla u) \tag{2.3}$$

This was the first step towards the development of a theory to semilinear approximations to FBPs. The paper [CJK] then suggests the study of the limit problem as $\varepsilon \to 0$. We studied this problem in [M1],[M2]. Here, F is a Lipchitz continuous function bounded away from 0 and infinity.

The strategy used to study the limit problem appears quite often in the theory of free boundary problems and resembles, at least philosophically, the regularity theory developed by Ennio De Giorgi to treat the regularity theory of minimal surfaces. This strategy can be macroscopically divided in two major parts: (A) The geometric measure theory of the free boundary and (B) The classification of global profiles appearing in the blow-up analysis at the free boundary points. The main difficulties to implement this strategy for this case are the following

a) To implement (A), we need create suitable solutions u_ε for (2.3) for which the uniform geometric estimates on their level surfaces can be obtained. This is the point where the nonvariational character of the problem strikes with full force. FBPs of variational type are much more well understood than nonvariational ones. In the variational setting, many striking techniques were developed and are available in the so called Alt-Caffarelli-Friedman theory.
b) To implement (B) in view of the "nondivergence structure" of the right hand side, by the presence of the gradient, the typical integration by parts argument that has been developed in ([CLW1]) and systematically used in similar problems does not seem to work, creating very challenging difficulties for the classification of global profiles in the blow-up analysis.

The least supersolution approach is used to construct solutions satisfying geometric properties of the level surfaces that are uniform in ε. The key argument in part (A) is to recover a uniform linear growth rate away from the level surfaces $\{u_\varepsilon = \varepsilon\}$. This is performed by constructing specific barriers that are supersolutions and control the height decay of the least supersolutions in the interior of the domain. By a Harnack type argument, this information propagates inside the region $\{u_\varepsilon > \varepsilon\}$ and leads to the desired linear growth rate.

The last important ingredient from the geometric measure theoretical point of view is to prove that the reduced free boundary $(F(u_0))_{red}$ of a limit has full Hausdorff measure. Due to the nonvariational character of the problem, the results in the so called Alt-Caffarelli theory ([AC]) (for minimizers) are not applicable. The proof relies on the construction of a specific barrier with a certain decay close to the free boundary which is "tested" against the perimeter of the reduced free boundary by the De-Giorgi's Gauss-Green Theorem. This procedure provides $\mathcal{H}^{N-1}$-density of the reduced free boundary. This allows to prove that the free boundary of a limit has the "right" weak geometry, in the measure theoretic sense.

The classification of the global solutions appearing in the blow-up analysis at the free boundary points is performed by introducing a rather geometric method, reminiscent from the A.D. Alexandrov moving planes method, where moving planes are replaced by barriers with uniformly curved free boundaries. These barriers are carefully constructed and their free boundaries uniformly bent in ε by the use of Kelvin transforms in larger and larger spheres whose centers and radii go to infinity. Important points here are, the minimality of the least supersolutions and a type of strong maximum principle that forbids an interior touching of the barriers. Not only the result but the new techniques introduced to implement (A) and (B) are the main contributions of this work. This work can be thought as a regularizing approach to viscosity theory of FBPs developed in [C1, C2, C3].

The final Theorem is

Theorem 1. *Suppose F is Lipschitz continuous bounded away from 0 and ∞. Let $\{u_\varepsilon\}_{\varepsilon>0}$ be a uniformly bounded family of least viscosity supersolutions to (2.3). For any sequence $\{u_{\varepsilon_j}\}$, there exists a subsequence $\{u_{\varepsilon_{j_k}}\}$ converging locally uniformly in Ω to a Lipschitz function u_0 which is a viscosity solution to the following (FBP)*

$$\begin{cases} \Delta u &= 0 \quad \text{in } \Omega \setminus \{u>0\} \\ H_\nu(u_\nu^+) - H_\nu(u_\nu^-) &= M \quad \text{on } \Omega \cap \partial\{u>0\}, \end{cases} \tag{2.4}$$

with $H_\nu(t) = \int_0^t \frac{s}{F(s\nu)} ds$.
Furthermore, the free boundary $F(u_0) := \Omega \cap \partial\{u_0>0\}$ is a $C^{1,\alpha}$ surface around $\mathcal{H}^{N-1}$a.e. point.

We observe that in this case, the free boundary condition

$$H_\nu(u_\nu^+) - H_\nu(u_\nu^-) = M \text{ on } F(u)$$

also depends on the normal direction to the free boundary. These type of free boundary conditions appear as a limit of homogenization problems in periodic media. For homogenization FBPs, we refer to [CLM1], [CLM2].

Variational free boundary problems. Since the last decade, there has been an extensive effort to extend the existence and regularity theory of FBPs developed by Luis Caffarelli in [C1],[C2],[C3] to a wider class of elliptic partial differential equations. Prof. Sandro Salsa and his group in Italy have achieved great advancements in this theory. Among many important works in that direction, we could mention [CFS], [FS], [PW1, PW2].

In this spirit, Luis Caffarelli and Sandro Salsa suggested me and E. Teixeira to investigate the problem for the case of divergence form elliptic PDE via singular perturbation techniques. In this work [MT3], we used the regularizing methods approach to construct nonnegative solutions to the following FBP

$$\begin{cases} \operatorname{div}(A(x)\nabla u) &= 0 \quad \text{in } \Omega \setminus \partial\{u>0\} \\ u_\nu^2 &= \frac{2\Gamma}{\langle A\nu,\nu\rangle} \quad \text{on } F(u), \end{cases} \tag{2.5}$$

where $A = A(x)$ is a Holder continuous and uniform elliptic matrix and Γ a continuous positive function. We consider the following family of regularizing equations

$$\begin{cases} \operatorname{div}(A(x)\nabla u) &= \Gamma(x)\beta_\varepsilon(u) \quad \text{in } \Omega \\ u &= \varphi \quad \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

with

$$\Gamma \in C^0(\Omega),\ \inf_\Omega \Gamma = \mathcal{I} > 0,\ \varphi \in C^{1,\alpha}(\Omega),\ \partial\Omega \in C^{1,\alpha},\ \ \beta_\varepsilon \quad \text{as before.}$$

We solve (2.6) by considering minimizers of the following variational problem

$$\text{minimize } \ \{\mathcal{F}_\varepsilon(u) : u \in H^1_\varphi(\Omega)\} \tag{2.7}$$

where

$$\mathcal{F}_\varepsilon(u) = \int_\Omega \{\frac{1}{2}\langle A(x)\nabla u, \nabla u\rangle + \Gamma(x)B_\varepsilon(u)\}dx$$

and

$$B_\varepsilon(t) = \int_0^t \beta_\varepsilon(\tau)d\tau$$

Our main result in [MT3] is

Theorem 2. *Let u_0 be a limit of the minimizers $\{u_\varepsilon\}_{\varepsilon>0}$ considered above. Then u_0 is a viscosity solution to (2.5). Furthermore, if $A = A(x), \Gamma = \Gamma(x) \in C^{0,1}(\overline{\Omega})$, the free boundary $\partial\{u_0 > 0\} \cap \Omega$ is a $C^{1,\gamma}$ surface around $\mathcal{H}^{N-1}$ a.e. point.*

In this work we prove the uniform Lipschitz regularity of minimizers by using Harnack inequality arguments and a kind of Hopf Lemma barriers that allows us to estimate the gradient of u_ε along level surfaces. The linear uniform growth rate away from the ε level surfaces is obtained. The level surfaces $\{u_\varepsilon = C\varepsilon\}$ (C universal) are proven to converge in the Hausdorff distance to the free boundary of the limit. Among the main contribution of this work is the extension of previous techniques for the case of low regularity of the coefficients $A(x) \in C^\alpha$. We observe that all results in [AC] except the regularity (A is assumed to be Lipschitz just for that matter) of the free boundary are extended to the case C^α.

2-B. Hausdorff measure estimates for free boundaries of nonlinear elliptic equations

One of the fundamental goals in the theory of free boundary problems is to obtain a description as accurate as possible of the geometry and the regularity of the interface that separates the phases of the system in consideration. At least from the philosophical point of view, we can divide the free boundary smoothness results in two categories: "measure theoretic or weak" and "higher order or strong" results.

Measure theoretic results mean that from the point of view of the geometric measure theory, the interface is a "nice" surface. In mathematical terms, if $0 \in F(u)$ is on the free boundary of u, this amounts to say that $F(u)$ has locally a finite Hausdorff measure with estimate of the following type

$$\mathcal{H}^{N-1}(F(u) \cap B_r(0)) \leq Cr^{N-1} \tag{2.8}$$

Non-degeneracy properties of solutions in free boundary problems are a key ingredient to study the regularity of the interface. Non-degeneracy means heuristically that solutions grow at the maximum admissible rate dictated by the regularity of the problem in question. These fundamental properties guarantees that global profiles of the blow-up analysis are nontrivial and information about the free boundary is kept by taking limits. For the particular case of the estimate (2.8), non-degeneracy make a bridge between the energy type estimates of the solutions along the free boundary and the measure of its neighborhood. See for example [AC],[C3],[C5].

We now state the simplest version of an important result due to Luis Caffarelli.

Theorem 3 ([AC],[C3])**.** *Let $u \in C^{0,1}(\overline{B_1})$. Assume that $u \geq 0$ and*

a) $\Delta u = 0$ *in* $\{u > 0\}$
b) $u(x) \geq Cd(x, F(u))$ *for* $x \in B_{1/2}$, *where* $0 \in F(u) := \partial\{u > 0\} \cap B_1$. *Then, if* $x_0 \in B_{1/3} \cap F(u)$,

$$\mathcal{H}^{N-1}(F(u) \cap B_r(x_0)) \leq \overline{C}r^{N-1}$$

In other words, the theorem states that linear growth rate away from the free boundary for Lipschitz nonnegative harmonic functions implies a rather restrictive geometry of the free boundary. The result above was known to hold for divergence form operators with Holder continuous coefficients [C3] and also for the case where the free boundary is replaced by some level surfaces as in [CS] and [MT3].

A crucial step in the direction of the proof of the result above is the strong-nondegeneracy lemma

Theorem 4 ([C3])**.** *Under the conditions of Theorem (3), u is strong non-degenerate, i.e,*

$$\sup_{B_r(0)} u \geq Dr \quad \textit{for} \;\; 0 \leq r \leq \frac{1}{4}$$

The proof of this result is very ingenious and uses the construction of a polygonal sequence of points where u has a uniform gain in its values along the points of the sequence. This uniform gain is proven by using a kind of averaging process of the values of u for points close to the free boundary by the means of the Poisson kernel representation of the solution.

An interesting result in the direction of the estimate (2.8) was proven by Pei-Yong Wang in [PW3]. There, he replaces a) in theorem (3) by $F(D^2u) = 0$ in $\{u > 0\}$ and $F(M) = \inf_{\alpha\in\mathcal{A}} L_\alpha(M)$ where $\mathcal{A}$ is a family of indices and each L_α is a uniformly elliptic operator with constant coefficients. We observe that in this case F is concave and the Poisson kernel representation technique can also be used for operators L_α.

During my post-doc period at the University of Iowa, Lihe Wang and I became interested in investigating inhomogeneous free boundary problems for nonlinear elliptic operators. To our knowledge, very few results are known for the inhomogeneous case beyond [CJK1]. In particular, we worked on the extension of theorem (3) for fully nonlinear operators and quasi-linear operators of $p-$Laplace type, where the previous techniques are not available. By using a technique based on the De-Giorgi oscillation lemma, we are able to prove the strong non-degeneracy property for the subsolution class $\underline{S}(\lambda, \Lambda, f)$ introduced by Luis Caffarelli in [Ca1]. (See also [CC]). All the following results are in ([MW1]).

Proposition 5. *Let $u \in C^{0,1}(\overline{B_1})$. Assume that $u \geq 0$ and*

a) $u \in \underline{S}(\lambda, \Lambda, f)$ *in* $\{u > 0\}$, *where* $f \in L^q(B_1),\ q > N$.
b) $u(x) \geq Cd(x, F(u))$ *for* $x \in B_{1/2}$, *where* $0 \in F(u) := \partial\{u > 0\} \cap B_1$. *Then,*

$$\sup_{B_r(0)} u \geq Dr \quad \text{for} \ \ 0 \leq r \leq \frac{1}{4}$$

Furthermore, the same result is true if we replace a) by

$a')$ $\Delta_p u \geq f(x)$ *in the sense of distributions, where* $f \in L^q(B_1),\ q > N$.

This result allows us to extend theorem (3) to subsolutions with the following structural conditions

Theorem 6. *Let $u \in C^{0,1}(\overline{B_1})$. Assume that $u \geq 0$ and F satisfy the following structural conditions*

i) *F(M,x) is uniformly elliptic with constants* $0 < \lambda < \Lambda$ *and* $F(0, x) \in L^q(B_1), q > N$.
ii) *Suppose there exists a matrix* $A(x) \in C^{1,1}(B_1)$ *and* $B(x)$ *a function in* $L^q(B_1), q > N$ *such that*

$$L_A(M) = \mathit{trace}(A(x)M) + B(x) \geq F(M, x) \text{ for all } (x, M) \in B_1 \times \mathcal{S}^{n\times n}$$

a) $F(D^2u, x) \geq f(x)$ *in* $\{u > 0\}$, *where* $f \in L^q(B_1),\ q > N$.
b) $u(x) \geq Cd(x, F(u))$ *for* $x \in B_{1/2}$, *where* $0 \in F(u) := \partial\{u > 0\} \cap B_1$. *Then, if* $x_0 \in B_{1/3} \cap F(u)$,

$$\mathcal{H}^{N-1}(F(u) \cap B_r(x_0)) \leq \overline{C}r^{N-1}$$

Furthermore, the conditions a) can replaced by

$a')$ $\Delta_p u \geq f(x)$ *in the sense of distributions, where* $f \in L^q(B_1),\ q > N$.

Assumption a) replaces concavity of F by the existence of a $C^{1,1}$ family of planes that are above the graph of F within a L^q distance to the graph. We remark that in the above results we can replace the free boundary by some level surfaces as in [CS] and [MT3]. As an application of the results above, we can study the Hausdorff measure of a singular perturbation problem governed by the $p-$Laplace operator with forcing terms in $L^q, q > N$. These models appear in the theory of flame propagation in the description of laminar flames when sources are present (see, [LW1]). We consider the regularizing family of elliptic equations in a smooth bounded domain Ω.

$$\Delta_p u = \beta_\varepsilon(u) + f_\varepsilon \text{ in } \Omega \tag{2.9}$$

Let me state the simplest version of our results

Theorem 7. *Let u_ε be a family of nonnegative solutions of (2.9) and β_ε as in (2.1). Assume that $||u_\varepsilon||_{L^\infty(\Omega)} \leq L_1$ and $||f_\varepsilon||_{L^\infty(\Omega)} \leq L_2$. Then, for every compact set $K \subset\subset \Omega$ there exists a constant $C = C(p, N, L_1, L_2, \beta)$ independent of ε such that*

$$||\nabla u_\varepsilon||_{L^\infty(K)} \leq C$$

The laplacian case ($p = 2$) and f_ε uniformly bounded was proven in [LW1], by Caffarelli-Jerison-Kenig monotonicity formula in [CJK]. The one phase case when $f_\varepsilon \equiv 0$ was proven by A. Petrosyan, D. Danielli and H. Shahgholian in [DPS] using a nice compactness argument. Our proof is direct and we use the strategy adopted in [CS] by using Harnack inequality argument and a type of Hopf lemma barrier to control the gradient of solutions along level surfaces. We show that variational solutions of the equation (2.9) have a uniform growth rate from the ε level surfaces. This way, proceeding like in [MT3], we can prove the following

Theorem 8. *Let u_ε be variational solutions of (2.9) and assume $||u_\varepsilon||_{L^\infty(\Omega)} \leq L_1$ and $||f_\varepsilon||_{L^\infty(\Omega)} \leq L_2$. Then, for any sequence $\varepsilon_n \to 0$ the exists a subsequence $\varepsilon_n^{'} \to 0$ and a function $f \in L^\infty(\Omega)$ such that $\Delta_p u = f$ in $\{u > 0\}$ and $\mathcal{H}^{N-1}(F(u) \cap B_r(x_0)) \leq \overline{C} r^{N-1}$.*

We also extend the above results for the case where the assumption $||f_\varepsilon||_{L^\infty(\Omega)} \leq L_2 < \infty$ is replaced by $||f_\varepsilon||_{L^q(\Omega)} \leq L_2, q > N$. This involves $C^{1,\alpha}$ boundary regularity theory and a version of a nonlinear Hopf Lemma is small scales for quasilinear equations of $p-$Laplace type.

2-C. Boundary behavior of solutions to nonlinear elliptic PDEs at regular points

In this section, I describe my latest results with Lihe Wang about the boundary behavior of elliptic equations. The understanding of how solutions to elliptic equations approach the boundary of the domain where they are defined is a central question in the field. The regularity theory for solutions near the boundary is strongly related with the geometry and regularity of the boundary itself. This way, difficult questions frequently appear when information about the regularity and geometry of the boundary is not known a priori. This is the case in the theory free of boundary problems (FBPs).

A bedrock result in the theory of elliptic PDEs is the Hopf Lemma. It has profound consequences that range from the strong maximum principle to estimates for solutions and regularity theory. A rough geometric description of this result says that a positive harmonic functions that vanishes in a piece of the boundary that have an inner touching ball hits the boundary with a nontrivial angle.

A deep refinement of the result was proven by Luis Caffarelli (see theorem (9) below) where he shows a definite asymptotic behavior (differentiability) for harmonic function that are Lipschitz up to the boundary. This result is of paramount importance in the theory of FBPs. During the period I have been in The University of Iowa, Lihe Wang and I have worked to extend this result to inhomogeneous fully nonlinear elliptic equations in domains with low regularity.

In the sequel, I state Luis Caffarelli's result in [C2] and the extensions we have obtained.

Theorem 9 (Caffarelli, [C2])**.** *Let u be a positive harmonic function in Ω such that*

a) *$0 \in \partial\Omega$, $u \equiv 0$ in $B_1 \cap \partial\Omega$, and at 0 there exists an inner touching B ($B \subset \Omega$ with $\{0\} = \overline{B} \cap \partial\Omega$).*
b) *$u \in C^{0,1}(\overline{\Omega \cap B_1})$. Then, if ν is the inner normal to B at 0,*

$$u(x) = \alpha\langle x, \nu\rangle^+ + o(|x|) \quad \text{in } B \text{ with } \alpha > 0$$

If B is an outer touching ball ($B \subset \Omega^C$ with $\{0\} = \overline{B} \cap \partial\Omega$) and if ν is the inner normal to B at 0

$$u(x) = \alpha\langle x, \nu\rangle^- + o(|x|) \quad \text{in } B^C \text{ with } \alpha \geq 0$$

Furthermore, in this last case if $\alpha > 0$ the ball is tangent to $\partial\Omega$.

Let me point out a simple version of a result we obtained for fully nonlinear case in ([MW2])

Theorem 10. *Let u be a nonnegative in Ω such that*

a) $0 \in \partial\Omega$, $u \equiv 0$ *in* $B_1 \cap \partial\Omega$, *and at* 0 *there exists an inner touching* B ($B \subset \Omega$ *with* $\{0\} = \overline{B} \cap \partial\Omega$).
b) $u \in C^{0,1}(\overline{\Omega \cap B_1})$ *and* $u \in \overline{S}(\lambda, \Lambda, f)$ *where* $f \in L^q, q > N$.

Then, if ν is the inner normal to B at 0,

$$u(x) = \alpha\langle x, \nu\rangle^+ + o(|x|) \quad \text{in } B \text{ with } \alpha \geq 0 \ (\alpha > 0 \text{ if } f \equiv 0) \tag{2.10}$$

c) *In the case, B is an outer touching ball ($B \subset \Omega^C$ with $\{0\} = \overline{B} \cap \partial\Omega$) and if ν is the inner normal to B at 0*

$$u(x) = \alpha\langle x, \nu\rangle^- + o(|x|) \quad \text{in } B^C \text{ with } \alpha \geq 0$$

Furthermore,in this last case if $\alpha > 0$ the ball is tangent to $\partial\Omega$.

Actually, we can relax the condition Lipschitz regularity in *b*) by asking instead $u \in S(\lambda, \Lambda, f)$. In this case, as long as u does have a infinite normal derivative at 0 the asymptotic development in (2.10) will hold in nontangential regions inside B. Also, the general form of the theorem holds in the case where the regular points on the boundary are of the $C^{1,\alpha}$ type. This means that we can replace the touching ball inside (or outside) the domains by a $C^{1,\alpha}$ touching surface.

With this machinery in hand, we are able to define classes of sub and supersolutions $\underline{S}(\lambda, \Lambda, C, G)$ and $\overline{S}(\lambda, \Lambda, C, G)$ (analogous to the class $\underline{S}(\lambda, \Lambda, C)$ and $\overline{S}(\lambda, \Lambda, C)$ introduced by Luis Caffarelli in [Ca1], [CC]) of free boundary problems with elliptic free boundary condition $G(u_\nu^+, u_\nu^-, \nu) = 0$ and show as in [ACS1] the equivalence of definitions of solutions to FBPs when the are defined by comparison principle or by asymptotic development.

The ingredients involved to prove these results use constructions of boundary barriers, inhomogeneous and nonlinear version of Hopf Lemma and Krylov boundary estimates for the case where $f \in L^\infty$ and Lihe Wang boundary estimates for $C^{1,\alpha}$ domains for the case where $f \in L^q, q > N$.

Lihe Wang and I are currently working on extending the above results for more general boundary points like $C^{1,dini}$ regular points. These would require to understand Hopf type lemmas and gradient boundary estimates for these type of domains. This amounts to generalize the $C^{1,\alpha}$ estimates obtained in Lihe Wang's Ph.D thesis ([W1]) for $C^{1,dini}$ boundaries and boundary data. Also, by the techniques employed in [MW1], it is likely that these results can be extended for operators of $p-$Laplace type. Some results have already been obtained ([MW3]).

2-D. Future Directions and Projects

In this section I will describe some of the projects and topics of research I plan to be involved with in the future.

1) FBPs for operators in divergence form with C^α coefficients. Since the last decade, there has been an extensive effort to extend the regularity theory developed by Luis Caffarelli in ([C1, C2, C3]) to a wider class of elliptic partial differential equations. Many important advancements have been accomplished by important works of Sandro Salsa, Fausto Ferrari, Cristina Ceruti, Mikhail Feldman, Pei Yong Wang ([CFS], [FS],[Fe1], [Fe2], [PW1],[PW2], [PW3]), by extending the results for nondivergence elliptic equations with C^α coefficients and divergence operator with Lipschitz coefficients. Recently, John Lewis and Kaj Nystrom in [LN] provide a very nice improvement for the $p-$Laplace case. Despite of all these developments, the case of divergence operator with C^α coefficients remains untreated. I would like to study this question in detail. It is interesting to observe that this theory would automatically provide the regularity for the FBP in [MT3] with C^α coefficients. This is a project together with Luis Caffarelli and Lihe Wang.

2) Symmetry for elliptic equations in half-space. The study of the free boundary problem in [M1], motivates to investigate bounded positive solutions in a half-space $\mathbb{R}^N_+ = \{x_N > 0\}$ of the following equation $\Delta u = \beta(x_n, u)F(|\nabla u|)$ and $u = 0$ on $\{x_N = 0\}$. Here $\beta : [0, \infty] \times [0, M] \to \mathbb{R}$ is continuous, Lipschitz in the second variable and F is say Lipschitz bounded away from 0 and ∞. I would like to study symmetry properties of the solutions, in particular, if solutions depend on just one variable and are strictly increasing. This result can be thought as an analogue of the Gidas-Ni-Nirenberg symmetry theorem for solutions in a ball. This would extend the theory developed in [BCN1]. The moving "spheres" method and comparison principles with ODEs developed in [M1] can very likely be applied to successfully investigate this case.

3) Tangential touch between fix and free boundaries. In this project, I would like to investigate properties related to the tangential touch between the fixed boundary and the free boundary. Recently, it was show in [KaKS] that for Alt-Caffarelli-Friedman minimizers under the appropriate conditions of the boundary data, the free boundary touches the fixed boundary in a tangential fashion. It would be of interest to develop an equivalent result in the nonvariational setting. More specifically, to study the tangential touch properties for the least supersolution constructed in [C3].

4) Convexity / Non-convexity of level surfaces for semilinear equations. In the beautiful work of H. Shahgholian and R. Monneau, [MS], they prove the following interesting result

Theorem 11. *There exists $\Omega^+ \subset \Omega^- \subset \mathbb{R}^2$ two nested convex domains and a smooth function f satisfying*

$$f \leq 0, \quad \text{in } (-\infty, +\infty), \;\; f = 0 \quad \text{in } (-\infty, -1) \cup (0, +\infty)$$

and a solution of

$$\begin{cases} \Delta u &= f(u) & \text{in } \mathcal{R} = \Omega^- \setminus \Omega^+ \\ u &= -1 & \text{on } \partial\Omega^- \\ u &= +1 & \text{on } \partial\Omega^+ \end{cases} \tag{2.11}$$

having a nonconvex level set. Moreover every bounded solution to (2.11) has at least one non-convex level set.

The proof of the Theorem relies essentially in two ingredients: (1) An example constructed by Acker in [Ack] and a singular perturbation free boundary problems of the type $\Delta u = \beta_\varepsilon(u)$. Based on the recent developments for more general singular perturbations FBPs, it would be interesting to investigate this lack of convexity for level surfaces for equations of the type $F(D^2u) = g(u)H(\nabla u)$, for nontrivial functions H (i.e, $H \neq 1$).

5) Full regularity of free boundary problems in divergence form. This problem was suggested to me by Prof. David Jerison (MIT). In [CJK2], it was proven that the free boundary of the minimizers of the functional of the following type $J(u) = \int |\nabla u|^2 + \chi_{\{u>0\}}$ is fully regular in dimensional $N = 3$. The 2-dimensional case is known since mid 80's in [ACF]. The idea now is to investigate the same type of result for functionals of the type $Z(u) = \int \langle A(x)\nabla u, \nabla u\rangle + \chi_{\{u>0\}}$ at least in the case where A is positive definite and Lipschitz continuous. For the nonvariational counterpart, the full regularity for the least supersolution developed in [C3] is also an interesting open question to be investigated even in the 2 dimensional case.

6) $W^{2,p}$ up to the boundary estimates for fully nonlinear equations. In the paper [WG-WL], $W^{2,p}$ estimates up to the boundary were obtained for solutions of the heat equation in domains with some assumptions involving the mean curvature. In this projet with Lihe Wang, we are interested to think about the extension of these estimates for fully nonlinear equations in the case where we could weaken the conditions for the mean curvature to reproduce the estimates. We are interested in curvature conditions in a weaker sense that could be described in the viscosity theory setting or in the variational sense introduced by Umberto Massari in [GM].

7) Boundary Harnack inequality in Reifenberg flat domains. In [SY], it was proved the doubling character of caloric functions in Lipschitz domains. The method introduced there can treat divergence and nondivergence equations simultaneously. This is a joint project with Lihe Wang. We are interested to investigate the same type of properties - boundary Harnack inequality - for elliptic PDEs in divergence and nondivergence form with bounded measurable coefficients in Reifenberg flat domains.

8) Free boundary problems with curvature. In [ACKS], the following variational-type problem is considered: Minimize

$$E(v,\Omega) = \int_D |\nabla v|^2 dx + \mathrm{Per}(\Omega),$$

where $D \subset \mathbb{R}^n$ is a bounded smooth domain, and the minimizer belongs to a class of admissible pairs (v,Ω) with respect to a fixed function $g \in H^1(D)$. Here $\mathrm{Per}(\Omega)$ denotes the perimeter of Ω. After showing the existence of a minimizer (u,Ω), the authors study regularity properties of u under various conditions, and in particular they obtain some properties of the level set $S = \{u = 0\}$. If $\kappa(S)$ is the mean curvature of the level set $\{u = 0\}$, the free boundary condition for this problem is $\kappa(S) = (u_\nu^+)^2 - (u_\nu^-)^2$. This provided a natural framework to create variational solutions for FBPs involving curvature. It would be very interesting to develop a nonvariational approach for these type of problems using techniques in [C3] and [M1]. Also, I would be interested to investigate regularity results for the free surface of the problem when the functional E is replaced by

$$E(v,\Omega) = \int_D \langle A(x)\nabla u, \nabla u\rangle dx + \mathrm{Per}(\Omega),$$

with minimal regularity on the coefficient A (say C^α).

3. Harmonic Analysis

3-A. Bilinear Maximal Operator in Sobolev Spaces

Regularity of maximal operators. For $f : \mathbb{R}^n \to \mathbb{R}$ locally integrable, the classical Hardy-Littlewood maximal operator is defined as

$$Mf(x) = \sup_{R>0} \frac{1}{m(B_R)} \int_{B_R(x)} |f(y)| \,\mathrm{d}y, \tag{3.1}$$

where $B_R(x)$ denotes the ball of radius R centered in x and $m(A)$ denotes the n-dimensional Lebesgue measure of the measurable set $A \subset \mathbb{R}^n$. For $\Omega \subset \mathbb{R}^n$ a proper open subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ we can define the local maximal operator at a point $x \in \Omega$ by

$$M_\Omega f(x) = \sup_{0<R<\delta_x} \frac{1}{m(B_R)} \int_{B_R(x)} |f(y)| \,\mathrm{d}y, \tag{3.2}$$

where the supremum is taken over all radii R such that $0 < R < \delta_x := \mathrm{dist}(x, \partial\Omega)$. Over the last decade there has been considerable interest in understanding the regularity properties of maximal operators, for instance how the weak differentiability is preserved. The first work in this direction is due to Kinnunen ([Ki]) in 1997 when he observed that the classical Hardy-Littlewood maximal operator is bounded in the Sobolev space $W^{1,p}(\mathbb{R}^n)$ for $p > 1$, using functional analytic tools (weak compactness arguments). Later developments on the subject include the boundedness of the local maximal operator in $W^{1,p}(\Omega)$, $p > 1$ ([KL]), and the continuity of the maximal operator in $W^{1,p}(\mathbb{R}^n)$, $p > 1$ ([Lu]). Other interesting papers related to this topic are [AP], [HO] and [KS].

In this joint work with Emanuel Carneiro ([CM1]), we studied this theory for other classes of maximal operators. We considered the following family of bilinear maximal operators in $\mathbb{R}^n$. For $\alpha \neq 1$ set

$$\mathcal{M}(f,g)(x) = \sup_{R>0} \frac{1}{m(B_R)} \int_{B_R} |f(x-\alpha y) g(x-y)| \,\mathrm{d}y. \tag{3.3}$$

An application of Hölder's inequality tells us that this operator maps $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ into $L^r(\mathbb{R}^n)$ where $1/p+1/q=1/r$, $1<p,q<\infty$ and $r>1$. In 2000, M. Lacey in the remarkable paper [L] showed that the family of one-dimensional bilinear maximal operators defined by (3.3) maps $L^p(\mathbb{R})\times L^q(\mathbb{R})$ into $L^1(\mathbb{R})$ where $1/p+1/q=1$, $1<p,q<\infty$, solving a conjecture posed by A. Calderón in 1964. This opened the way for us to prove the following result.

Theorem 12. *Given $\alpha\neq 1$, the bilinear maximal operator $\mathcal{M}$ defined in (3.3) maps $W^{1,p}(\mathbb{R}^n)\times W^{1,q}(\mathbb{R}^n)\to W^{1,r}(\mathbb{R}^n)$ boundedly and continuously, where $1/p+1/q=1/r$, $1<p,q<\infty$ and*

(a) *$r\geq 1$, if $n=1$;*
(b) *$r>1$, if $n>1$.*

Boundedness is a consequence of the following pointwise estimate

$$|\nabla\mathcal{M}(f,g)(x)|\leq \mathcal{M}(f,|\nabla g|)(x)+\mathcal{M}(|\nabla f|,g)(x)\quad a.e.\ x\in\mathbb{R}^n.$$

Because of Lacey's theorem, the case $n=1$ and $r=1$ becomes the key difference between the bilinear maximal and the prior works on the classical Hardy-Littlewood maximal operator. The functional analytic arguments in [Ki] and [KL], relying on the reflexivity of $L^r(\mathbb{R}^n)$ for $r>1$, are no longer available. To overcome this difficult, we adopted an approach introduced in [HO]. For the continuity part we followed the insightful and elegant proof of Luiro in [Lu]. Again the case $n=1$, $r=1$ is a new feature.

We also study the behavior of the almost everywhere and weak convergence under the action of the classical Hardy-Littlewood maximal operator, both in its global and local versions. Issues about the stability of the weak convergence under nonlinear operators are very interesting and have been studied in [MT1], [MT2] for a certain class called Nemytskii nonlinearities. Given an operator $T:E\to F$ between Banach spaces and $u_k\rightharpoonup u$ in E, the question is whether or not we have $T(u_k)\rightharpoonup T(u)$ in F (in the affirmative case for all such sequences $\{u_k\}_{k\geq 1}$, we say that T is sequentially weakly continuous.) We show the following.

Theorem 13. *For the global and local maximal operators defined in (3.1) and (3.2) we have*

(i) *$M:L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)$ and $M_\Omega:L^p(\Omega)\to L^p(\Omega)$, for $p>1$, are not sequentially weakly continuous.*
(ii) *If Ω is a bounded domain with Lipschitz boundary, then $M_\Omega:W^{1,p}(\Omega)\to W^{1,p}(\Omega)$ is sequentially weakly continuous for $p>1$.*
(iii) *$M:W^{1,p}(\mathbb{R}^n)\to W^{1,p}(\mathbb{R}^n)$, for $1<p<\infty$ is sequentially weakly continuous.*

3-B. Multilinear Maximal Operators

This is a joint work Emanuel Carneiro [CM2]. Here, we prove for a general family of fractional maximal multilinear operators, that L^p-boundedness implies $W^{1,p}$-boundedness. This provides applications to previous results on maximal multilinear operators obtained by M. Lacey [L], C. Kenig and E. Stein [KeSt], L. Grafakos and N. Kalton [GK], and C. Demeter, T. Tao and C. Thiele [DTT]. We now introduce some notation to formulate our result.

Let $f_j:\mathbb{R}^n\to\mathbb{R}$, $j=1,2,...,k$, be k measurable functions. Let l and m be natural numbers, and consider a family of $n\times m$ real matrices A_{ij} indexed by $i=0,1,2,...,l$ and $j=1,2,...,k$. Let α be a real number. With this set of parameters in hand, we define the following fractional maximal multilinear operator

$$\begin{aligned} &M_{A,\alpha,k,l,n}(f_1,f_2,...,f_k)(x)\\ &\quad=\sup_{r>0}\frac{r^\alpha}{m(B_r)^l}\int_{B_r\times B_r\times\ldots\times B_r}\prod_{j=1}^{k}\left|f_j\Big(A_{0j}x+\sum_{i=1}^{l}A_{ij}x_i\Big)\right|\mathrm{d}x_1\,\mathrm{d}x_2\ldots\mathrm{d}x_l\,, \end{aligned}\tag{3.4}$$

where $x,x_1,x_2,...,x_l\in\mathbb{R}^n$, B_r is the ball centered at the origin with radius r in $\mathbb{R}^n$, and $m(B_r)$ is the n-dimensional Lebesgue measure of this ball. Henceforth, we will assume the following condition

$$\sum_{j=1}^{k} \frac{1}{p_j} - \frac{\alpha}{n} = \frac{1}{q}\,, \tag{3.5}$$

We prove the general principle that, for the family of operators (3.4), an L^p-bound implies a $W^{1,p}$-bound with a pointwise estimate for the gradient. Our result is the following

Theorem 14. *Let $1 \leq p_1, p_2, ..., p_k, q \leq \infty$ be such that*

$$M_{\alpha,k,l,m,n} : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \times ... \times L^{p_k}(\mathbb{R}^n) \to L^q(\mathbb{R}^m)$$

is a bounded operator. Then

$$M_{\alpha,k,l,m,n} : W^{1,p_1}(\mathbb{R}^n) \times W^{1,p_2}(\mathbb{R}^n) \times ... \times W^{1,p_k}(\mathbb{R}^n) \to W^{1,q}(\mathbb{R}^m)$$

is also a bounded operator and the following pointwise estimate is holds a.e. x in $\mathbb{R}^N$,

$$\left|\nabla M(\mathbf{f})(x)\right| \leq \sum_{j=1}^{k} |A_{0j}|\, M(f_1, ..., f_{j-1}, |\nabla f_j|, f_{j+1}, ..., f_k)(x)$$

where $\mathbf{f}(x) = (f_1, f_2, ..., f_k)(x)$ and $M := M_{A,\alpha,k,l,n}$

3-C. Future Projects

Project 1. It is believed that the bilinear maximal operator in $\mathbb{R}^n$, $n > 1$, also maps $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ if $1/p + 1/q = 1$, $1 < p, q < \infty$ (M. Lacey, personal communication). If this is indeed the case, we can include $r = 1$, $n > 1$ in Theorem 12 above.

Project 2. In the paper [HO], Hajłasz and Onninen raised the following question: is the operator $f \mapsto |\nabla Mf|$ bounded from $W^{1,1}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$? H. Tanaka in [Tan] gave a very elegant positive proof in the case $n = 1$ for the non-centered maximal operator. I am interested in proving (or disproving) the analogous of Tanaka's result for the centered maximal operator.

Project 3. I believe the study of the centered maximal operator in the spaces $BV[a,b]$ or $BV(\mathbb{R})$ is a very interesting topic. One interesting question is if these spaces remain invariant under the action of the centered maximal operator. For the centered maximal operators, almost any advancement in theses lines will be a new result.

Project 4. Non-centered maximal operators in some circumstances actually improve the regularity of some class of Sobolev functions. This was proven by Kinnunen and Saksman [KS]. I believe it is an interesting question to investigate this improvement of regularity for more general mutli-linear maximal operators of the form $M_{A,\alpha,k,l,n}$.

4. Nonlinear Functional Analysis

The weak continuity of operators in Banach spaces is an important topic in the applications of functional analysis. This issue becomes more complex in a nonlinear enviroment. Usually, this type of questions arises naturally when one considers some nonlinear elliptic PDEs with critical sobolev exponents and some variational problems presenting lack of compactness for minimizing sequences. Typical examples of such cases can be found, for instance, in the Brezis-Nirenberg results in [BrN] and concentration and compactness theory developed by Lions in [Li].

Motivated by these type of questions, in this joint work with Eduardo Teixeira [MT1], [MT2], we studied the stability of the weak convergence under Nemytskii nonlinearites acting on vector valued Lebesgue spaces. These nonlinearities are often found in variational problems, non-linear Poisson equations and differential geometry. Some applications are then given to study a nonlinear weak spectral problem in $W^{1,p}$, as well as, proof simplifications for some theorems in linear theory.

More concretely, let $\mathbb{E}$ and $\mathbb{F}$ are Banach spaces and $f : \Omega\times\mathbb{E} \to \mathbb{F}$ a caratheodory function. Assume the Nemytskii operator $u \mapsto N_f(u)$ where $N_f(u)(x) = f(x, u(x))$ maps $L^p(\Omega,\mathbb{E})$ to $L^q(\Omega,\mathbb{F})$, $1 \le p, q \le \infty$. We define the problem $\alpha_{p,q}$ to be:

$\alpha_{p,q}$— Does N_f maps almost everywhere and weakly convergent sequences in $L^p(\Omega,\mathbb{E})$ into weakly convergent sequences in $L^q(\Omega,\mathbb{F})$?

The idea is to study the relations among the exponents p, q and the functional analytic properties of $\mathbb{E}$ and $\mathbb{F}$ to obtain an answer for the problem $\alpha_{p,q}$ above. Using these ideas, we prove the following weak spectral result in $W^{1,p}$. Let me present one of the results we have obtained.

Let $1 < p < \infty$ and Ω be domain in $\mathbb{R}^N$. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be such that:

(1) For a.e. $x \in \Omega$, $f(x, \cdot)$ is a Lipschitz function and $\sup_{x\in\Omega} \|f(x, \cdot)\|_{\mathrm{Lip}} = L < \infty$.
(2) $f(\cdot, t) \in W^{1,p}(\Omega)$, and $\|f(\cdot, t)\|_{W^{1,p}} \le C,\ \forall t \in \mathbb{R}$.

Given a $\psi \in W^{1,p}(\Omega)$ and $\lambda \in \mathbb{R}$ we are interested in finding a $u \in W^{1,p}(\Omega)$ such that

$$f(x, u(x)) - \lambda u(x) = \psi(x) \text{ a.e. } x \in \Omega. \tag{4.1}$$

Theorem 15. *Equation (4.1) is solvable for all $\lambda > L$. Moreover the solution is unique and the resolvent operator $(N_f - \lambda Id)^{-1} : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ is sequentially weakly continuous.*

We also present new proofs of the Brezis-Lieb Theorem [BrL], where we extended it to the Lebesgue vector valued spaces and the Zolezzi result in [Zo] about weak convergence in $L^\infty(\Omega)$ among others results.

4-A. Future Projects

Weak continuity of maximal and rearrangement operators. I would be interested to investigate issues related to the weak continuity and continuity of the class of multilinear maximal operators $M_{\alpha,k,l,m,n}$ defined in the previous section. Weak continuity of various types of rearrangement operators of harmonic analysis seems also to be an interesting topic to pursue.

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Department of Mathematics, University of Iowa, Iowa City, IA, 52242.
E-mail address: dmoreira@math.uiowa.edu
URL: www.math.uiowa.edu/~dmoreira