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Calculus of Variations

A singular perturbation free boundary problem for elliptic equations in divergence form

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Abstract In this paper we study the free boundary problem arising as a limit as $\varepsilon \to 0$ of the singular perturbation problem $\operatorname{div}(A(x)\nabla u) = \Gamma(x)\beta_\varepsilon(u)$, where $A = A(x)$ is Holder continuous, β_ε converges to the Dirac delta δ_0. By studying some suitable level sets of u_ε, uniform geometric properties are obtained and show to hold for the free boundary of the limit function. A detailed analysis of the free boundary condition is also done. At last, using very recent results of Salsa and Ferrari, we prove that if A and Γ are Lipschitz continuous, the free boundary is a $C^{1,\gamma}$ surface around $\mathcal{H}^{N-1}$ a.e. point on the free boundary.

1 Introduction

In this paper we are concerned about studying the limit as $\varepsilon \to 0$ of the solutions u_ε of the following Dirichlet problem

$$\begin{cases} \operatorname{div}\big(A(x)\nabla u\big) = \Gamma(x)\beta_\varepsilon(u) & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω and $\varphi \geq 0$ are sufficiently smooth, Γ is strictly positive, bounded and continuous, the coefficient matrix $A = A(x)$ is Holder continuous and β_ε is an approximation of Dirac δ_0 measure in the sense that

$$\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta\left(\frac{s}{\varepsilon}\right), \tag{1.2}$$

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where

(1) $\beta \in C^{\infty}(\mathbb{R})$, support of β lies in $[0,1]$ and it is positive in $(0,1)$.

(2) $\int_0^1 \beta(s)\mathrm{d}s := 1$.

This problem is a interesting model in combustion and flame propagation theory. It appears in the description of laminar flames as an asymptotic limit for high energy activation.

The idea is that Eq. (1.1) approximates the free boundary problem

$$\operatorname{div}\big(A(x)\nabla u\big) = 0 \quad \text{in } \ \Omega^+ := \{x \in \Omega : u(x) > 0\} \tag{1.3}$$

and

$$\langle A\nabla u, \nabla u\rangle = 2\Gamma \text{ along } \partial\{u > 0\} \cap \Omega \tag{1.4}$$

in a certain sense that will be discussed later on. Actually, there are several ways to express the free boundary condition. Another way could be

$$u_\nu^2 = \frac{2\Gamma}{\langle A\nu, \nu\rangle} \text{ along } \partial\{u > 0\} \cap \Omega.$$

where ν is the inward unit normal to the free boundary $\partial\{u > 0\} \cap \Omega$.

It is a fruitful idea to study free boundary problems like Eq. (1.3) via regularizing problems like (1.1). See for example [13] Information about the original free boundary problem can often be obtained by establishing results for the approximating ones that are uniform on ε. This will be the approach of this paper. We will study geometric properties of the limit function and its free boundary by establishing the same properties (estimates) for the approximating funtions u_ε and its suitable level sets that will approach the free boundary of the limit in the Hausdorff distance.

We should point out that, although the setting involves a Dirichlet problem with sufficiently smooth boundary data we are essentially interested in interior regularity results. For this reason, the concerns in this paper will be of local nature. Boundary issues could also be treated by following the ideas developed in [3]. The main goals of this paper are the development of ε-uniform estimates for the approximating perturbed problems, the description of in what senses the free boundary condition (1.4) is satisfied and the free boundary regularity in the case where A is Lipschitz continuous.

Our paper is organized as follows: In Sect. 2, a detailed formulation of the problem is carried out. Since, geometric properties like linear growth away from the free boundary and nondegeneracy are expected to hold, we restringe ourselves to deal with more “stable" solutions of the problem (1.1), namely, the minimizers of the variational problem associated. The core of this section is the proof of the uniform Lipschitz estimates for the approximating functions u_ε. Section 3 will be in charge of the main uniform in ε geometric properties of the approximating functions. It is proven in this section the linear growth away from the ε level sets, strong nondegeneracy and measure estimate of neighborhoods of those level sets. Section 4 is devoted to the limit problem. The limit function is proven to be the minimizer of the correspondent minimization free boundary problem and all the previous geometric properties are now proven for the limit case. The uniform density and the measure totality of the reduced free boundary is obtained. Section 5, the last one, is devoted to the study of the free boundary condition. This condition is proven to hold in four weak senses, namely, integral sense, measure sense, pointwise sense and viscosity sense.

Last Theorem in Sect. 5 addresses the regularity of the free boundary. We use very recent results in the theory of regularity of free boundaries developed by Fausto Ferrari and Sandro Salsa to prove that under the assumption that $A = A(x)$, $\Gamma = \Gamma(x)$ are Lipschitz continuous, the free boundary is a $C^{1,\gamma}$ surface around $\mathcal{H}^{N-1}$ a.e. point on the free boundary. We are thankful to Fausto Ferrari and Sandro Salsa for sharing with us their latest results and for very interesting discussions about this topic. Finally, the regularity of the free boundary when $A = A(x)$ is just Holder continuous seems to be an important and delicate question. The authors expect to return to this issue in a future research.

2 Uniform Lipschitz regularity of solutions

In this section we shall explore some geometric properties of solutions of

$$\begin{cases} Lu = \operatorname{div}(A(x)\nabla u) = \Gamma(x)\beta_\varepsilon(u) & \text{in } \Omega \quad \text{(PDE)} \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where hereafter, we assume

$$\Gamma \in L^\infty(\Omega), \quad \inf_\Omega \Gamma = \mathcal{I} > 0$$

and for some fixed $\alpha \in (0,1)$, $\Lambda \geq \lambda > 0$,

$$A = A(x) \in C^\alpha(\overline{\Omega}), \quad \lambda I \leq A(x) \leq \Lambda I, \quad \varphi \in C^{1,\alpha}(\overline{\Omega}), \quad \partial\Omega \in C^{1,\alpha}$$

At this point, solutions are understood as weak (distributional) solutions, i.e, $u_\varepsilon \in H^1_\varphi(\Omega) = \{u \in H^1(\Omega) : u - \varphi \in H^1_0(\Omega)\}$ such that

$$\int_\Omega (\langle A(x)\nabla u, \nabla\varphi\rangle + \Gamma(x)\beta_\varepsilon(u)\varphi)\,\mathrm{d}x = 0 \quad \forall \varphi \in C_0^\infty(\Omega)$$

Remark 2.1 In this paper, the operator Lu will be always related to the matrix A above specified. If the context requires, the dependence of the matrix will be explicitly emphasized by writting

$$L_B(u) = \operatorname{div}(B(x)\nabla u)$$

Universal constants are constants depending on dimension N and ellipticity constants λ, Λ. The perturbation $\{\beta_\varepsilon\}_{\varepsilon>0}$ is given by (1.2) and we define B_ε to be

$$B_\varepsilon(u) = \int_0^u \beta_\varepsilon(s)\mathrm{d}s = \int_0^{u/\varepsilon} \beta(s)\mathrm{d}s.$$

In general, the problem (2.1) has multiple solutions since the comparison principle is not available. For this reason, we take advantage of the divergence structure of the above PDE to deal with particular solutions of the problem (2.1), namely, the minimizers of the following variational problem

$$\text{minimize } \left\{\mathcal{F}_\varepsilon(u) : u \in H^1_\varphi(\Omega)\right\} \tag{2.2}$$

where

$$\mathcal{F}_\varepsilon(u) := \int_\Omega \left\{ \frac{1}{2} \langle A(x)\nabla u, \nabla u \rangle + \Gamma(x) B_\varepsilon(u) \right\} \mathrm{d}x$$

For the sake of completeness we state the next Theorem. The proof is a straighfoward application of standart minimization arguments, elliptic regularity theory and maximum principle.

Theorem 2.2 *For each $\varepsilon > 0$, the variational problem (2.2) has minimizer $u_\varepsilon \in C^{1,\alpha}(\overline{\Omega})$. Furthermore, u_ε solves (2.1) in weak sense and $u_\varepsilon \geq 0$ in Ω.*

With previous Theorem in mind, when reffering to the solution of the ε-perturbed problems (2.1), we always mean the minimizers u_ε provided by Theorem above.

Remark 2.3 Since the minimizers satisfy $Lu_\varepsilon \geq 0$ in $\mathcal{D}'(\Omega)$, then Lu_ε is a Radon measure given by

$$\int_\Omega \phi L u_\varepsilon = -\int_\Omega \langle A(x)\nabla u_\varepsilon(x), \nabla \phi \rangle \, \mathrm{d}x \quad \text{for all } \phi \in H_0^1(\Omega)$$

This way, since the vector field $A(x)\nabla u_\varepsilon \in C^\alpha(\overline{\Omega})$ the Green's formula holds, i.e, for all $\phi \in C^0(\Omega) \cap H^1(\Omega)$ and $B_\rho(x_0) \subset\subset \Omega$, we have:

$$\int_{B_\rho(x_0)} \phi L u_\varepsilon + \int_{B_\rho(x_0)} \langle A(x)\nabla u_\varepsilon, \nabla\phi \rangle \, \mathrm{d}x = \int_{\partial B_\rho(x_0)} \phi \, \langle A(x)\nabla u_\varepsilon, \nu \rangle \, \mathrm{d}\mathcal{H}^{N-1}$$

Remark 2.4 (Lipschitz renormalization) Let $u \in H^1(B_r(x_0)) \cap C^0(\overline{B_r(x_0)})$ be a weak solution of

$$Lu = \Gamma(x)\beta_\varepsilon(u)$$

Define $w\colon B_{r/\varepsilon}(0) \to \mathbb{R}$, as

$$w(y) = \frac{1}{\varepsilon} u(x_0 + \varepsilon y),$$

then, by Change of Variables Theorem, $w \in H^1(B_{r/\varepsilon}(0)) \cap C^0(\overline{B_{r/\varepsilon}(0)})$ is a weak solution of

$$L^\varepsilon w = L_{A^\varepsilon}(w) = \Gamma(x_0 + \varepsilon y)\beta_1(w)$$

where

$$A^\varepsilon(y) = A(x_0 + \varepsilon y), \quad y \in B_{r/\varepsilon}(0)$$

Furthermore, if u is differentiable at 0, then $\nabla w(0) = \nabla u(x_0)$. It is interesting to notice that the renormalization above does not change the ellipticity constants of the operator L. This means, that this renormalization does not affect the constants appearing in Schauder estimates as well as in Harnack Inequality. The family of operators L^ε can be regarded as just one operator L.

Lemma 2.5 *Let* $w \in C^{1,\alpha}(B_1(0)) \cap H^1(B_1(0))$ *be a solution to*

$$Lw = g(x), \quad ||g||_{L^\infty(B_1(0))} \leq D$$

if $w(0) \leq 1$, *then there exists a universal constant* $C_0 > 0$ *such that*

$$|\nabla w(0)| \leq C_0$$

Proof By interior Schauder estimate, we have

$$|\nabla w(0)| \leq \overline{C}(||w||_{L^\infty(B_{1/2}(0))} + D)$$

and using Harnack inequality,

$$||w||_{L^\infty(B_{1/2}(0))} \leq C(1 + D|B_1(0)|^{1/n})$$

since $\overline{C}, C, D$ are universal, this concludes the lemma. □

Now, we state and prove a useful Lemma by itself.

Lemma 2.6 *Let* $0 \in \partial\Omega$ *and* $v \in C^{1,\alpha}(\Omega \cap B_1(0)) \cap H^1(\Omega \cap B_1(0))$ *be a nonnegative solution of*

$$Lv = 0 \quad in\ \Omega \cap B_1(0)$$

Suppose also that on $\Upsilon = \partial\Omega \cap B_1(0), v = 0$ *and* $|\nabla v|$ *is bounded. Then, there exist a universal constant* $C > 0$ *such that*

(1) $v(x) \leq Cdist(x, \partial\Omega) \sup_{\Upsilon} |\nabla v|$
(2) $||\nabla v||_{L^\infty(B_{1/2}(0))\cap\Omega)} \leq C \sup_{\Upsilon} |\nabla v|$

Proof Let us start proving item (1). Consider $x_0 \in B_{1/2}(0)\cap\Omega$. Setting $h = dist(x_0, \partial\Omega)$, $M = \sup_{\Upsilon} |\nabla v|$, $v(x_0) = \tau h$, we need to show that $\tau \leq CM$ for some universal constant $C > 0$. Reescaling v with respect to the distance h, i.e,

$$w(y) = \frac{1}{h} v(x_0 + hy)$$

Then, $w(0) = \tau$ and from remark (2.4), w is a nonnegative solution of

$$L_{A^h}(u) = 0 \text{ in } B_1(0), \text{ where } A^h(y) = A(x_0 + hy).$$

Since $h = dist(x_0, \partial\Omega)$, there exists $y_1 \in \partial B_1(0)$ where $w(y_1) = 0$. Moreover, since $\nabla w(y) = \nabla v(x_0 + hy)$, $y \in B_1(0)$, we have $|\nabla w(y_1)| \leq M$. Now, by Harnack inequality in $B_{1/2}(0)$, there exists a universal constant c such that

$$w \geq c\tau \text{ in } B_{1/2}(0).$$

The idea now is to construct a barrier touching w from below at y_1. Let Z^h be the a funtion such that

$$\begin{cases} L_{A^h}(Z^h) = 0 & \text{in } R \\ Z^h = 1 & \text{on } \partial B_{1/2}(0) \\ Z^h = 0 & \text{on } \partial B_1(0). \end{cases} \tag{2.3}$$

It is clear that, $Z^h \geq 0$, $Z^h \in C^{1,\alpha}(\overline{R})$ where $R = B_1(0) \setminus B_{1/2}(0)$.

Defining now,

$$Z_h(y) = c\tau Z^h(y),\ y \in \overline{R},$$

then

$$Z_h = 0 \le w \text{ on } \partial B_1(0) \quad \text{and} \quad Z_h \le c\tau \le w \text{ on } \partial B_{1/2}(0),$$

this way, we have

$$L_{A^h} w = L_{A^h} Z_h \quad \text{in } R \ \text{ and } \quad w \ge Z_h \text{ on } \partial R,$$

by the weak maximum principle,

$$w \ge Z_h \quad \text{in } R \ \text{ and } \quad w(y_1) = Z_h(y_1).$$

In particular,

$$\partial_\nu w(y_1) \ge \partial_\nu Z_h(y_1),$$

where ν is the unit inner normal to $B_1(0)$ at y_1. Since A is Holder continuous, Hopf's Lemma applies [12, Chap. 3], if $\delta_h = \inf\limits_{\partial B_1(0)} \partial\nu Z^h$

$$\partial_\nu Z_h \ge c\tau\delta_h > 0.$$

From the $C^{1,\alpha}$ estimate up to the boundary we can conclude

$$\inf_{0\le h\le 1/2} \delta_h \ge \delta > 0,$$

where δ is independent of h. Then

$$M \ge\ |\nabla w(y_1)| \ge \partial_\nu w(y_1) \ge c\tau\delta,$$

proving the first part of the Lemma. The second part now follows from Schauder estimate and Harnack inequality. Indeed, if we define $v^\star(y) := v(x_0 + hy),\ \ y \in B_1(0)$, then

$$|\nabla w(0)| \le \overline{C}\frac{1}{\mathrm{dist}(x_0,\partial\Omega)}||v^\star||_{L^\infty(B_{1/2}(0))} \le \frac{\overline{C}C}{\mathrm{dist}(x_0,\partial\Omega)}v(x_0) \le CA.$$

□

Now we are ready to proof the uniform Lipschitz estimate. Before, It is convenient to introduce the following notation that will simplify the statements of the Theorems in this and next sections,

$$\begin{aligned} B^\star_\alpha &= B_{\delta_\varepsilon}(x_\varepsilon) \text{ where } u_\varepsilon(x_\varepsilon) = \alpha \ \text{ and } \ \delta_\varepsilon = \frac{1}{2}\mathrm{dist}(x_\varepsilon,\partial\Omega) \\ \Omega_\alpha &= \{x \in \Omega;\ 0 \le u_\varepsilon(x) \le \alpha\} \ \text{ and } \ d_\alpha(x) = \mathrm{dist}(x,\Omega_\alpha) \\ \Omega^+_\alpha &= \Omega \setminus \Omega_\alpha = \{x \in \Omega; \quad u_\varepsilon(x) > \alpha\} \end{aligned}$$

Hereafter, we consider,

$$\Omega^{'} \subset\subset \Omega,\ \ \Delta = \mathrm{dist}(\Omega^{'},\partial\Omega)$$

Theorem 2.7 (Uniform Lipschitz estimate) *There exists a universal constant* $C = C(\Omega^{'})$ *such that for* $\varepsilon > 0$ *small enough*

$$||\nabla u_\varepsilon||_{L^\infty(\Omega')} \leq C$$

In particular, the family $\{u_\varepsilon\}_{\varepsilon>0}$ *is locally uniformly Lipschitz.*

Proof Let $x_0 \in \Omega^{'}$ and consider $\varepsilon < \Delta = \text{dist}(\Omega^{'}, \partial\Omega)$. The proof will be based in obtaining pointwise gradient estimates depending on where x_0 is placed at.

Case I $x_0 \in \Omega_\varepsilon$

Since $B_\varepsilon(x_0) \subset \Omega$ we can consider the Lipschitz renormalization $w = w_\varepsilon$ of u_ε in that ball,

$$w(y) = \frac{1}{\varepsilon} u_\varepsilon(x_0 + \varepsilon y), \quad y \in B_1(0).$$

By remark (2.4), $\nabla w(0) = \nabla u_\varepsilon(x_0)$ and w is a solution of

$$L_{A^\varepsilon}(w) = \Gamma(x_0 + \varepsilon y)\beta_1(w) \text{ in } B_1(0)$$

it follows now from Lemma (2.5) that

$$|\nabla w(0)| \leq C_0$$

where C_0 is a universal constant.

Case II $x_0 \in \Omega_\varepsilon^+$ and $d_\varepsilon(x_0) < \Delta/3$.

From the hypothesis, we have that $3d_\varepsilon(x_0) < \text{dist}(x_0, \partial\Omega)$. Let $y_0 \in \partial\Omega_\varepsilon$ such that $d_\varepsilon(x_0) = |x_0 - y_0|$ and $d = \text{dist}(y_0, \partial\Omega)$. Now, let us define the Lipschitz renormalization of u_ε in $\Omega_\varepsilon^+ \cap B_d(y_0)$, that is,

$$w(y) = \frac{u_\varepsilon(y_0 + dy) - \varepsilon}{d}, \quad y \in B_1(0) \cap D_\varepsilon$$

where $D_\varepsilon = T^{-1}(\Omega_\varepsilon^+), T(y) = y_0 + dy$.
Since $w = 0$ and $|\nabla w|$ is bounded along $B_1(0) \cap \partial D_\varepsilon$, by case I, and

$$L_{A^d}(w) = 0 \text{ in } B_1(0) \cap D_\varepsilon \text{ where } A^d(y) = A(y_0 + dy), \quad y \in D_\varepsilon.$$

we can use lemma (2.6) to conclude

$$||\nabla w||_{L^\infty(B_{1/2}(0)\cap D_\varepsilon)} \leq C$$

The estimate above, translated in terms of u_ε is equivalent to

$$||\nabla u_\varepsilon||_{L^\infty(B_{d/2}(y_0)\cap\Omega_\varepsilon^+)} \leq C$$

To conclude this case, it is enough to show that, $x_0 \in B_{d/2}(y_0)$. This follows since,

$$2|x_0 - y_0| = 2d_\varepsilon(x_0) < \text{dist}(x_0, \partial\Omega) - d_\varepsilon(x_0) \leq \text{dist}(y_0, \partial\Omega) = d.$$

Case III $x_0 \in \Omega_\varepsilon^+$ and $d_\varepsilon(x_0) \geq \Delta/3$.

In this case, since

$$Lu = 0 \text{ in } \Omega_\varepsilon^+$$

and

$$\Omega^{''} = \Omega^{'} \cap \{x \in \Omega;\ d_\varepsilon(x) \geq \Delta/3\} \subset\subset \Omega_\varepsilon^+, \text{ with } \operatorname{dist}(\Omega'', \partial\Omega_\varepsilon^+) \geq \frac{\Delta}{3}$$

Then using interior Schauder estimate, there exists a universal $C > 0$ such that

$$||\nabla u_\varepsilon||_{L^\infty(\Omega'')} \leq C$$

where $C = C(\frac{\Delta}{3})$, concluding the proof of the Theorem. □

3 Geometric properties of the level sets

In this section, we prove important geometric properties of the level sets. The idea is to obtain those properties independently of the particular u_ε considered. This way, we can prove that those properties hold for the free boundary of the limit function. In particular, this will imply a rather restrictive geometry of the free boundary.

Since we are interested in studying the geometric properties of the level sets, our analysis will be focused in regions close enough to the level sets of the minimizers u_ε.

Theorem 3.1 (Nondegeneracy) *Given $C_1 > 1$, there exists a universal constant C_2 depending on C_1 such that, if $x_0 \in B_\varepsilon^\star$ and $u_\varepsilon(x_0) \geq C_1\varepsilon$, then*

$$u_\varepsilon(x_0) \geq C_2 d_\varepsilon(x_0)$$

Proof Let us call $d = d_\varepsilon(x_0)$ and suppose $u_\varepsilon(x_0) = \lambda = \alpha d$. We want to show $\alpha \geq \underline{c} > 0$, for some contant $\underline{c}$. To this end, let us make a Lipschitz renormalization

$$w(y) := \frac{1}{d} u_\varepsilon(x_0 + dy).$$

From Remark (2.4), since $B_d(x_0) \subset \Omega_\varepsilon^+$, w satisfies

$$L_{A^d}(w) = 0$$

where

$$A^d(y) = A(x_0 + dy), \quad y \in B_1(0)$$

Furthermore $w(0) = \alpha$. By Harnack inequality,

$$\underline{c}\alpha \leq w \leq \overline{c}\alpha \quad \text{in } B_{1/2}.$$

Now, let ψ be a cut-off function such that $\psi \equiv 0$ in $B_{1/4}, \psi \equiv 1$ in $B_1 \setminus B_{1/2}$ and define

$$\zeta = \begin{cases} \min(w, \overline{c}\alpha\psi) & \text{in } B_{1/2} \\ w & \text{in } B_1 \backslash B_{1/2}. \end{cases}$$

Notice that, in the same way that we proceed in Remark (2.4), by using the Change of Variables Theorem, w minimizes the functional

$$\mathcal{E}(\xi) := \int\limits_{B_1} \left\{ \frac{1}{2}\langle A^d(y)\nabla\xi(y), \nabla\xi(y)\rangle + \Gamma(x_0 + dy)B_\varepsilon(d\xi(y)) \right\} \mathrm{d}y,$$

restricted to $K = w + H_0^1(B_1(0))$. Hence, since $\zeta \in K$ and w is a minimizer we have:

$$\mathcal{E}(\zeta) \geq \mathcal{E}(w).$$

Writing this down, we find

$$A \geq B, \tag{3.1}$$

where,

$$A := \int\limits_{B_1} \frac{1}{2}\Big(\langle A^d(y)\nabla\zeta(y), \nabla\zeta(y)\rangle - \langle A^d(y)\nabla w(y), \nabla w(y)\rangle\Big)\mathrm{d}y$$

and

$$B := \int\limits_{B_1} \Gamma(x_0 + \mathrm{d}y)\Big(B_\varepsilon(\mathrm{d}w) - B_\varepsilon(\mathrm{d}\zeta)\Big)\mathrm{d}y.$$

We estimate A by

$$\begin{aligned} A &= \frac{1}{2} \int\limits_{B_{1/2}\cap\{\overline{c}\alpha\psi \leq w\}} \Big(\overline{c}^2\alpha^2\langle A^d(y)\nabla\psi(y), \nabla\psi(y)\rangle - \langle A^d(y)\nabla w(y), \nabla w(y)\rangle\Big)\mathrm{d}y \\ &\leq \overline{c}^2\Lambda\alpha^2\|\nabla\psi\|_{L^2(B_1(0))} \\ &= C\alpha^2. \end{aligned}$$

Now we turn our attention to estimate B by below. Notice, initially that $w \geq \zeta$ and B_ε is a nondecreasing function, hence

$$\begin{aligned} B &\geq \int\limits_{B_{1/4}} \Gamma(x_0 + \mathrm{d}y)\Big(B_\varepsilon(\mathrm{d}w) - B_\varepsilon(\mathrm{d}\zeta)\Big)\mathrm{d}y \\ &= \int\limits_{B_{1/4}} \Gamma(x_0 + \mathrm{d}y)B_\varepsilon(\mathrm{d}w)\mathrm{d}y \\ &\geq \mathcal{I}\int\limits_{B_{1/4}} B_\varepsilon(\mathrm{d}w)\mathrm{d}y \\ &\geq \mathcal{I}|B_{1/4}|B_\varepsilon(\underline{c}\lambda) \\ &\geq \mathcal{I}|B_{1/4}|B_\varepsilon(\underline{c}C_1\varepsilon) \\ &= \mathcal{I}|B_{1/4}|B_1(C_1\underline{c}) = c^*. \end{aligned}$$

Putting those inequalities together we finish the proof. □

Corollary 3.2 (Linear growth away of the level set) *There exists a constant* $C = C(\Omega')$, *independent of* ε *such that*

$$C_2 d_\varepsilon(x_0) \leq u_\varepsilon(x_0) \leq C d_\varepsilon(x_0)$$

if

$$x_0 \in \Omega' \cap \{u_\varepsilon \geq C_1\varepsilon\}, \quad d_\varepsilon(x_0) \leq \frac{\Delta}{4}$$

Proof *The first inequality follows from the nondegeneracy above, just observing that if $d_\varepsilon(x_0) < \frac{\Delta}{3}$, then $x_0 \in B_\varepsilon^\star$, like case II in the proof of Theorem (2.7). Applying the same Theorem, Uniform Lipschitz estimate, we deduce the second inequality.* □

We turn our attention to a strong nondegeneracy result. Before we state the nondegeneracy condition, we prove an interesting lemma by itself. It is a variation of Lemma 7 presented in [6]. In this situation, we treat the case where there exists a linear growth away from level set δ if we are confined to a large multiple of that level set .

Lemma 3.3 (*Strong Nondegeneracy Lemma*) *Assume that $v \geq 0$ is a Lipschitz solution of:*

$$Lv = 0 \textit{ in } \Omega \cap B_R(\xi) \textit{ such that} \tag{3.2}$$

(1) $v \equiv \delta$ *on* $\partial\Omega \cap B_R(\xi), 0 \in \partial\Omega$
(2) $v(x_0) \geq C\delta > 0,\ C >> 1$ *with* $x_0 \in B_{R/2}(\xi)$
(3) $v(x) \geq D \cdot dist(x, \partial\Omega)$ *in* $\{v \geq C\delta\} \cap B_{R/2}(\xi)$

Then, there exists a universal constant $M = M(C, D, Lip(v))$ such that

$$\sup_{B_r(x_0)} v \geq Mr \quad \textit{for} \quad 0 < r \leq \frac{R}{4}$$

Proof Let $B_\rho(x_0)$ be the largest ball contained in $\{v > \delta\}$. Consider $y_0 \in \partial B_\rho(x_0)$ such that $\rho = |x_0 - y_0| = \text{dist}(x_0, \partial\Omega)$ with $v(y_0) = \delta$. From the assumptions, $B_\rho(x_0) \subset \Omega \cap B_R(\xi)$. By (3),

$$v(x_0) \geq D \cdot \rho$$

Now, by Lipschitz continuity, taking $h = \frac{D}{C\text{Lip}(v)}$ we have:

$$v \leq \left(\frac{1}{C} + \frac{\text{Lip}(v)h}{D}\right) v(x_0) \leq \frac{v(x_0)}{2} \quad \text{in } B_{\rho h}(y_0)$$

Therefore,

$$v \leq \frac{v(x_0)}{2} \quad \text{in } A = B_{\rho h}(y_0) \cap \partial B_\rho(x_0)$$

Notice that $\mathcal{H}^{N-1}(A)$ depends only on the fixed constant h. This way, it is independent of the particular points x_0, y_0 considered above. Proceeding as in Lemma 7 in [6], we obtain $0 < \overline{C}(\mu, h) < 1$ such that

$$\sup_{\partial B_\rho(x_0)} v \geq \left[1 + \overline{C}(\mu, h)\right] v(x_0)$$

In particular, we can find $x_1 \in \partial B_\rho(x_0)$ such that,

$$v(x_1) \geq (1 + \overline{C}) v(x_0)$$

and

$$v(x_1) - v(x_0) \geq \overline{M}|x_1 - x_0| \quad \text{where } \overline{M} = \overline{C}D$$

The idea is to construct a polygonal along which v grows linearly, starting from x_0. For that, we repeat the same construction done in Theorem 1.9 in [7] and this will finish the proof. □

Theorem 3.4 (Strong nondegeneracy) *There exists $C = C(\Omega')$ such that*

$$\sup_{B_\rho(x_0)} u_\varepsilon \geq C\rho, \quad \text{for } \rho \leq \frac{\Delta}{12}$$

if

$$x_0 \in \Omega' \cap \{u_\varepsilon \geq C_1\varepsilon\}, \quad d_\varepsilon(x_0) \leq \frac{\Delta}{6}, \;\; C_1 \gg 1$$

Proof All that we have to do is to verify that we are in the configuration of the previous Lemma. Indeed, if x_0 is in the above conditions, and x_ε is such that $d_\varepsilon(x_0) = |x_0 - x_\varepsilon|$ if we denote $\delta_\varepsilon = \operatorname{dist}(x_\varepsilon, \partial\Omega)$, we have:

$$\frac{2\Delta}{3} = \Delta - \frac{\Delta}{3} < \operatorname{dist}(x_0, \partial\Omega) - d_\varepsilon(x_0) \leq \delta_\varepsilon$$

Thus,

$$x_0 \in \frac{1}{2}\overline{B_{\frac{\Delta}{3}}(x_\varepsilon)} \subset B_{\frac{\Delta}{3}}(x_\varepsilon) \subset \frac{1}{2}B_{\delta_\varepsilon}(x_\varepsilon) = B^\star_\varepsilon \tag{3.3}$$

Besides,

$$\bigcup B_{\frac{\Delta}{3}}(x_\varepsilon) \subset \mathcal{N}_{\frac{\Delta}{2}}(\Omega') \subset\subset \Omega \tag{3.4}$$

Inclusion (3.3) joint with Theorem (3.1) says that we can use the previous Lemma with $B_R(\xi) = B_{\frac{\Delta}{3}}(x_\varepsilon)$ and inclusion (3.4) says that we can take uniform Lipschitz constant for all those balls, proving the Theorem. □

Hereafter, we will assume $C_1 >> 1$ such that the strong nondegeneracy property above holds.

Corollary 3.5 (Uniform positive density along level sets) *There exist universal constants $C_3 = C_3(\Omega') > 1$, $C_4 = C_4(\Omega')$ such that if $x_0 \in \Omega'$, $d_\varepsilon(x_0) \leq \frac{\Delta}{6}$ with*

$$u_\varepsilon(x_0) = \mu \geq C_1\varepsilon \quad \text{and} \quad C_3\mu \leq \rho \leq \frac{\Delta}{6},$$

then for $\mu, \varepsilon > 0$ small enough,

$$\frac{|B_\rho(x_0) \cap \{u_\varepsilon > \mu\}|}{|B_\rho(x_0)|} \geq C_4.$$

Proof By nondegeneracy above,

$$\sup_{B_{\rho/2}(x_0)} u_\varepsilon \geq C\rho$$

So, there exists $y_0 \in \overline{B_{\rho/2}(x_0)}$ such that

$$u_\varepsilon(y_0) \geq \frac{C\rho}{2}$$

Then, by Lipschitz continuity, we have for $\varepsilon > 0$ small enough

$$d_\varepsilon(y_0) \geq \frac{\widetilde{C}\rho}{2}$$

This way, choosing $C_4 > 0$ small enough such that $B_{\rho C_4}(y_0) \subset B_{\widetilde{C}\rho/2}(y_0) \cap B_\rho(x_0)$, by Harnack inequality, we find

$$u_\varepsilon(x) \geq \overline{C} u_\varepsilon(y_0) \geq \frac{\overline{C}C\rho}{2} \geq \frac{\overline{C}CC_3\mu}{2} \quad \text{in } B_{\rho C_4}(y).$$

Thus, if μ is small enough, we can choose, $C_3 > 1$ large enough such that

$$\frac{\overline{C}CC_3}{2} > 1$$

Allowing us to conclude

$$B_{\rho C_4}(y) \subset B_\rho(x_0) \cap \{u_\varepsilon > \mu\}.$$

This finishes the proof. □

Lemma 3.6 *Let* $x_0 \in \Omega' \cap \partial\Omega^+_{C_1\varepsilon}$, *and* $\mu > 3C_1\varepsilon$. *Then, there exists a constant* $C = C(\Omega')$ *such that for* $\rho \leq \dfrac{\Delta}{6}$

$$\int_{\{C_1\varepsilon < u_\varepsilon < \mu\} \cap B_\rho(x_0)} |\nabla u_\varepsilon|^2 \mathrm{d}x \leq C\mu\rho^{N-1}$$

Proof Indeed, by Green's formula in remark (2.3), applied with $\phi = \min\left\{(u_\varepsilon - C_1\varepsilon)^+, \mu - C_1\varepsilon\right\} \geq 0$, we have

$$\int_{B_\rho(x_0)} \phi L u_\varepsilon + \int_{B_\rho(x_0)} \langle A(x)\nabla u_\varepsilon, \nabla\phi\rangle \,\mathrm{d}x = \int_{\partial B_\rho(x_0)} \phi \,\langle A(x)\nabla u_\varepsilon, \nu\rangle \,\mathrm{d}\mathcal{H}^{N-1}$$

So,

$$\int_{\{C_1\varepsilon < u_\varepsilon < \mu\} \cap B_\rho(x_0)} \langle A(x)\nabla u_\varepsilon, \nabla\phi\rangle \,\mathrm{d}x \leq \int_{\partial B_\rho(x_0) \cap (\Omega^+_{C_1\varepsilon})} \langle \phi \cdot A(x)\nabla u_\varepsilon, \nu\rangle \,\mathrm{d}\mathcal{H}^{N-1}$$

By ellipticity and Lipschitz continuity, the Lemma follows. □

Now, we prove the last Lemma before studying the Hausdorff measure of the level sets.

Lemma 3.7 *Let* $x_0 \in \Omega' \cap \partial\Omega^+_{C_1\varepsilon}$, $d_\varepsilon(x_0) \leq \frac{\Delta}{16}$ *and* $\mu > 3C_1\varepsilon$. *There exists a universal constant* $C^\star = C^\star(\Omega')$ *such that if* $C^\star\mu \leq 2\rho \leq \dfrac{\Delta}{16}$ *then for* $\mu, \varepsilon > 0$ *small enough* ($\mu << \rho$), *we have:*

$$|\{C_1\varepsilon < u_\varepsilon < \mu\} \cap B_\rho(x_0)| \leq \overline{C}\mu\rho^{N-1}$$

where $\overline{C} = \overline{C}(\Omega')$

Proof Let us consider $\{B_j\}$ a finite overlapping convering of $\partial\Omega^+_{C_1\varepsilon} \cap B_{2\rho}(x_0)$ by balls of radius $C^\star\mu$ and center on $\partial\Omega^+_{C_1\varepsilon}$. We know, from the Heine–Borel Theorem that, the balls of this covering can overlap at most m times, m depending only on the dimension, i.e, $\sum \chi_{B_j} \le m(N)$. Let us define

$$w_\varepsilon = \min\left\{(u_\varepsilon - C_1\varepsilon)^+, \mu - C_1\varepsilon\right\} + C_1\varepsilon \text{ and } L \ge ||\nabla u_\varepsilon||_{L^\infty(\mathcal{N}_{\frac{\Delta}{8}}(\Omega'))}$$

Clearly,

$$w_\varepsilon(x) = \begin{cases} C_1\varepsilon, & 0 \le u_\varepsilon(x) \le C_1\varepsilon \\ u_\varepsilon(x), & C_1\varepsilon < u_\varepsilon(x) < \mu \\ \mu & u_\varepsilon(x) \ge \mu \end{cases} \tag{3.5}$$

It is easy to see that

$$\bigcup B_j \subset \mathcal{N}_{\frac{\Delta}{8}}(\Omega^{'}) \cap B_{4\rho}(x_0)$$

There exist B_j^1, B_j^2 subballs of B_j such that

(1) The radius of $B_j^1, B_j^2 \sim \mu$ (by constants depending on $\Omega^{'}$)
(2) $w_\varepsilon \ge \frac{3}{4}\mu$ in B_j^1 and $w_\varepsilon \le \frac{2}{3}\mu$ in B_j^2

Indeed, let us observe first that there exists $x_1 \in \frac{1}{4}\overline{B_j}$ so that

$$u_\varepsilon(x_1) = \sup_{\frac{1}{4}B_j} u_\varepsilon \ge C\frac{C^\star\mu}{4}, \quad C = C(\Omega^{'})$$

Now, if μ is small enough, we can choose $C^\star$ large enough such that

$$C^\star C > 4, \ L > \frac{1}{C^\star} \text{ where } L \ge ||\nabla u_\varepsilon||_{L^\infty(\mathcal{N}_{\frac{\Delta}{8}}(\Omega'))}$$

Setting $r_j^1 = \frac{1}{8L}\mu$ and $r_j^2 = \frac{1}{3L}\mu$ we have

$$u_\varepsilon \ge \frac{3}{4}\mu > \frac{1}{3}\mu > C_1\varepsilon \text{ in } B_j^1 = B_{r_j^1}(x_1)$$
$$u_\varepsilon \le \frac{2}{3}\mu < \mu \text{ in } B_j^2 = B_{r_j^2}(x_0)$$

From this and the definition of w_ε in (3.5), we prove the required properties for B_j^1 and B_j^2. Defining $m_j = \fint_{B_j} w_\varepsilon$, we have that $|w_\varepsilon - m_j| > D\mu$ in at least one of the two subballs B_j^1, B_j^2, where D is a universal constant. Indeed, otherwise, we could find sequences $x_n \in B_j^1, y_n \in B_j^2$ such that,

$$\frac{|w_\varepsilon(x_n) - m_j|}{\mu} < \frac{1}{n}, \quad \frac{|w_\varepsilon(y_n) - m_j|}{\mu} < \frac{1}{n} \quad \forall n$$

Providing

$$\frac{|w_\varepsilon(x_n) - w_\varepsilon(y_n)|}{\mu} \to 0$$

contradicting (2). Now, by Poincare's inequality, we have

$$\underline{D}^2\mu^2 \le ⨍_{B_j} |w_\varepsilon - m_j|^2 \mathrm{d}x \le \overline{D}(C^\star\mu)^2 ⨍_{B_j} |\nabla w_\varepsilon|^2 \mathrm{d}x$$

which implies

$$\int\limits_{B_j\cap\{C_1\varepsilon<u_\varepsilon<\mu\}} |\nabla u_\varepsilon|^2\mathrm{d}x = \int\limits_{B_j} |\nabla w_\varepsilon|^2\mathrm{d}x \ge D^\star |B_j|$$

By nondegeneracy, since $\mu << \rho$

$$B_\rho(x_0)\cap\{C_1\varepsilon < u_\varepsilon < \mu\} \subset \mathcal{N}_{\frac{\mu}{C_2}}(\partial\Omega^+_{C_1\varepsilon}\cap B_{2\rho}(x_0))$$

This way, if $C^\star$ is taken large enough, then

$$B_\rho(x_0)\cap\{C_1\varepsilon < u_\varepsilon < \mu\} \subset \bigcup 2B_j \subset B_{4\rho}(x_0)$$

yielding

$$\int\limits_{B_{4\rho}(x_0)\cap\{C_1\varepsilon<u_\varepsilon<\mu\}} |\nabla u_\varepsilon|^2\mathrm{d}x \ge \int\limits_{\cup 2B_j\cap\{C_1\varepsilon<u_\varepsilon<\mu\}} |\nabla u_\varepsilon|^2\mathrm{d}x$$

$$\ge \frac{1}{m}\sum \int\limits_{2B_j\cap\{C_1\varepsilon<u_\varepsilon<\mu\}} |\nabla u_\varepsilon|^2\mathrm{d}x \ge \frac{D^\star}{m}\sum|B_j| \ge \frac{D^\star}{m}|B_\rho(x_0)\cap\{C_1\varepsilon < u_\varepsilon < \mu\}|$$

Applying previous Lemma, we conclude this one. □

Theorem 3.8 *If* $x_0 \in \Omega' \cap \partial\Omega^+_{C_1\varepsilon}, d_\varepsilon(x_0) \le \frac{\Delta}{16}$, $\rho >> \mu > 3C_1\varepsilon$ *and* $C^\star\mu \le 2\rho \le \frac{\Delta}{16}$, *then*

$$\left|\mathcal{N}_\mu(\partial\Omega^+_{C_1\varepsilon})\cap B_\rho(x_0)\right| \le C_5\mu\rho^{N-1},$$

where $C_5 = C_5(\Omega')$ *is a universal constant.*

Proof Let us use the Corollary (3.5). Since $C_3 > 1$ we have

$$\mathcal{N}_\mu(\partial\Omega^+_{C_1\varepsilon}\cap B_\rho(x_0)) \subset \mathcal{N}_{C_3\mu}(\partial\Omega^+_{C_1\varepsilon}\cap B_\rho(x_0))$$

Now by that corollary, taking $\delta = C_3\mu$

$$\frac{\left|B_\delta(x)\cap\Omega^+_{C_1\varepsilon}\right|}{\left|B_\delta(x)\right|} \ge C_4 \quad \text{if } x\in\partial\Omega^+_{C_1\varepsilon}$$

This way, there exists $\underline{C} = \underline{C}(C_4, N)$ such that

$$|\mathcal{N}_\delta(\partial\Omega^+_{C_1\varepsilon})\cap B_\rho(x_0)| \le \frac{1}{2^N C_4}|\mathcal{N}_\delta(\partial\Omega^+_{C_1\varepsilon})\cap B_\rho(x_0)\cap\Omega^+_{C_1\varepsilon}| + \underline{C}\delta\rho^{N-1}$$

By Lipschitz continuity, there exist $D = D(C_3, \mathrm{Lip}(u_\varepsilon|\mathcal{N}_{\frac{\Delta}{8}}(\Omega'))$

$$\mathcal{N}_\delta(\partial\Omega^+_{C_1\varepsilon})\cap B_\rho(x_0)\cap\Omega^+_{C_1\varepsilon} \subset \{C_1\varepsilon < u_\varepsilon < D\mu\}\cap B_\rho(x_0)$$

Using, Lemma (3.7), since $\mu << \rho$

$$|\mathcal{N}_\delta(\partial\Omega^+_{C_1\varepsilon})\cap B_\rho(x_0)\cap\Omega^+_{C_1\varepsilon}| \le \overline{C}D\delta\rho^{N-1}$$

Concluding the Theorem. □

4 The limit problem

In this section we shall study the free boundary problem (1.3) by letting $\varepsilon \to 0$. Since $||u_\varepsilon||_{L^\infty(\overline{\Omega})} \leq \sup\limits_{\overline{line\Omega}} \varphi$ and the gradient, ∇u_ε, is locally bounded in Ω, there exists a subsequence $u_k = u_{\varepsilon_k}(\varepsilon_k \to 0)$ converging to u_0 in the following senses:

(1) $u_k \rightharpoonup u_0$ in $H^1_{\text{loc}}(\Omega)$
(2) $u_k \to u_0$ uniformly over compacts.

The function u_0 is our natural candidate to solve problem (1.3). We shall denote $\Omega_0 := \{x \in \Omega \bigm| u_0(x) > 0\}$ and $F(u_0) := \partial\Omega_0 \cap \Omega$. Again, throughout this section, $\Omega^{'} \subset\subset \Omega$ and $\Delta = dist(\Omega^{'}, \Omega)$.

Theorem 4.1 (Properties of the limit function u_0) *According to the above notation*

(1) $u_0 \in C^{0,1}_{loc}(\Omega)$, $\quad Lu_0 \geq 0$ *in* Ω *and*

$$Lu_0 = 0 \quad in\ \Omega_0.$$

Thus, Lu_0 *is a Radon measure supported on the free boundary and given by*

$$-\int\limits_\Omega \langle A(x)\nabla u_0, \nabla\varphi\rangle\, \mathrm{d}x = \int\limits_\Omega \varphi Lu_0 \quad \forall\varphi \in C_0^\infty(\Omega) \tag{4.1}$$

In Particular, $Lu_k \rightharpoonup Lu_0$ *in the sense of Radon measures.*

(2) u_0 *is strongly nondegenerate, that is, for any point* $x_0 \in \Omega^{'} \cap (\Omega_0 \cup F(u_0))$, $d(x_0, F(u_0)) \leq \frac{\Delta}{6}$ *there exists a constant universal constant* $C = C(\Omega^{'})$ *such that*

$$\sup_{B_\rho(x_0)} u_0 \geq C\rho \quad \text{for } \rho \leq \frac{\Delta}{12}$$

(3) Ω_0 *is the limit in the Hausdorff distance of* $\Omega_k = \{u_k > C_1\varepsilon_k\}$. *That is, given* $\delta > 0$, *for k large enough,*

$$\Omega^{'} \cap \Omega_k \subset \mathcal{N}_\delta(\Omega_0) \cap \Omega^{'}$$
$$\Omega^{'} \cap \Omega_0 \subset \mathcal{N}_\delta(\Omega_k) \cap \Omega^{'}$$

(4) (*Linear growth away of the free boundary*) *There exist universal constants* $\overline{C}$ *depending on* $\Omega^{'}$ *such that*

$$C_2 \text{dist}(x_0, \{u_0 = 0\}) \leq u_0(x) \leq \overline{C}\text{dist}(x_0, \{u_0 = 0\})$$
$$if \quad x_0 \in \Omega^{'} \cap \Omega_0, \ \text{dist}(x_0, \{u_0 = 0\}) \leq \frac{\Delta}{4}$$

where, C_2 *is the universal constant appearing in* (3.2).

(5) (*Uniform positive density along the free boundary*) *There exist a constant* $\tau = \tau(\Omega^{'})$ *such that for* $x_0 \in F(u_0) \cap \Omega^{'}$

$$\frac{|B_\rho(x_0) \cap \Omega_0|}{|B_\rho(x_0)|} \geq \tau \quad for\ \rho \leq \frac{\Delta}{6}$$

In particular, $|F(u_0)| = 0$

(6) *There exists a universal constant* $C = C(\Omega^{'})$ *such that*

$$\left|\mathcal{N}_\delta(F(u_0)) \cap B_\rho(x_0)\right| \le C\delta\rho^{N-1}$$

for $x_0 \in F(u_0) \cap \Omega^{'}$ *and* $\delta > 0$ *small enough* ($\delta << \rho << \Delta$). *In particular,*

$$\mathcal{H}^{n-1}(F(u_0) \cap B_\rho(x_0)) \le C\rho^{N-1}.$$

Proof From the uniform convergence and uniform Lipschitz continuity of u_ε we conclude immediately that u_0 is locally Lipschitz in Ω. Let $x_0 \in \Omega_0$ and $B_\rho(x_0) \subset\subset \Omega_0$. Since $u_k \to u_0$ uniformly in $B_\rho(x_0)$, there exists a $\delta > 0$ such that $u_k \ge \delta$ for k large enough. Besides, for such k's we can assume $\delta > \varepsilon_k$, since $\varepsilon_k \to 0$. This way,

$$Lu_k = 0 \text{ in } B_\rho(x_0)$$

Since $u_k \rightharpoonup u_0$ in $B_\rho(x_0)$, we conclude that for any $\varphi \in C_c^\infty(B_\rho(x_0))$

$$0 = \int\limits_{B_\rho(x_0)} \langle A(x)\nabla u_k, \nabla\varphi\rangle \,\mathrm{d}x \to \int\limits_{B_\rho(x_0)} \langle A(x)\nabla u_0, \nabla\varphi\rangle \,\mathrm{d}x$$

The expression of Lu_0, the fact that $Lu_0 \ge 0$ in $\mathcal{D}(\Omega^{'})$ and the weak convergence $Lu_k \rightharpoonup Lu_0$ follows from the local weak convergence of the gradients in H^1.

Let us check the strong nondegeneracy. Let $x_0 \in \Omega^{'} \cap \Omega_0$ and $y_0 \in F(u_0)$ such that $|x_0 - y_0| = d(x_0, F(u_0)) \le \frac{\Delta}{6}$. Assume $u(x_0) = \delta$. Clearly, Since $\varepsilon_k \to 0$, for k large enough $u(x_0) = \delta \ge C_1\varepsilon_k$. From the fact that $u(y_0) = 0$ and the pointwise convergence, for k sufficiently large, $u_k(y_0) < \varepsilon_k$. By continuity, there exists $y_k \in (x_0, y_0)$ such that $u(y_k) = \varepsilon_k$. So, $d_{\varepsilon_k}(x_0) < \frac{\Delta}{6}$. Now, by nondegeneracy, there is a universal constant $C = C(\Omega^{'})$ independent of k such that

$$\sup_{B_\rho(x_0)} u_k \ge C\rho \quad \text{for } \rho \le \frac{\Delta}{12}$$

Since, $u_k \to u_0$ uniformly in $B_\rho(x_0)$ we obtain

$$\sup_{B_\rho(x_0)} u_0 \ge C\rho \quad \text{for } \rho \le \frac{\Delta}{12}$$

If $x_0 \in F(u_0)$, we use the previous result. Indeed, let ρ be as specified. Taking $y_0 \in \partial B_{\rho/4}(x_0) \cap \Omega_0$. Taking $K = \mathcal{N}_{\frac{\Delta}{12}}(\Omega^{'})$ and applying (2) to K, there exists a universal constant $C = C(\Omega^{'})$ such that

$$\sup_{B_\rho(x_0)} u_0 \ge \sup_{B_{\rho/4}(y_0)} u_0 \ge C\frac{\rho}{4}$$

This proves (2). To prove (3), suppose the first inclusion is not true. Therefore, there would exist a sequence of points $\{x_k\}$ and a positive real number $\alpha > 0$, that we can assume $\alpha < \frac{\Delta}{6}$, satisfying

(a) $\operatorname{dist}(x_k, \Omega_0) \ge \alpha$.
(b) $x_k \in \Omega_k \cap \Omega^{'}$.
(c) $x_k \to x_0$ with $\operatorname{dist}(x_0, \Omega_0) \ge \alpha$.

From (c), we conclude that $u_0(x_0) = 0$, while $u_k(x_k) > C_1\varepsilon_k$. For k large enough, $u_k(x_0) < \varepsilon_k$ and $|x_k - x_0| \le \frac{\Delta}{8}$, so we can find $y_k \in (x_k, x_0)$ such that $u_k(y_k) = \varepsilon_k$ and so, $d_{\varepsilon_k}(x_k) \leq \frac{\Delta}{8}$. By strong nondegeneracy, we can find $z_k \in \overline{B_\rho(x_k)}$ such that

$$u_k(z_k) = \sup_{B_\rho(x_k)} u_k \geq C\rho \quad \text{for } \rho \leq \frac{\Delta}{12}, \;\; C = C(\Omega^{'}) > 0$$

Let us take $\rho = \frac{\alpha}{8}$. But, for $|x_k - x_0| < \rho$, we have $B_\rho(x_k) \subset B_{\frac{\alpha}{2}}(x_0)$. Finally, up to a subsequence, $z_k \to z_0 \in B_{\frac{\alpha}{4}}(x_0)$, and, since $u_k(z_k) \to u_0(z_0)$, we would conclude.

$$0 = \sup_{B_{\frac{\alpha}{2}}(x_0)} u_0 \geq u(z_0) \geq C\rho$$

a contradiction.

Similarly, we conclude the second inclusion. Indeed, suppose the inclusion does not hold. It means, there exists a sequence $x_k \in \Omega_0 \cap \Omega^{'}$ such that $\text{dist}(x_k, \Omega_k) \geq \alpha$, for some fixed $\alpha > 0$. In particular, $u_k \leq C_1\varepsilon_k$ in $B_{\frac{\alpha}{2}}(x_k)$. Passing to a subsequence, we can assume $x_k \to x_0$; then, for $|x_k - x_0| < \frac{\alpha}{8}$, we have $B_{\frac{\alpha}{8}}(x_0) \subset B_{\frac{\alpha}{2}}(x_k)$ and thus, $B_{\frac{\alpha}{8}}(x_0) \subset \Omega \setminus \Omega_0$, a contradiction.

Let us concentrate now on the linear growth property. In the conditions of item (4), let $u_0(x_0) = \delta$. For k large enough, $u_k(x_0) \geq \frac{\delta}{2} \geq C_1\varepsilon_k$. By arguing like above, we can conclude $d_{\varepsilon_k}(x_0) \leq \frac{\Delta}{8}$ and by linear growth away from the level sets, we have

$$u_k(x_0) \geq C_2 d_{\varepsilon_k}(x_0) \tag{4.2}$$

Let $y_k \in \partial\Omega_{\varepsilon_k}$ such that $|y_k - x_0| = d_{\varepsilon_k}(x_0)$. Passing to a subsequence, we can assume, $y_k \to y_0$. Since, $u_k(y_k) = \varepsilon_k$, then $u_0(y_0) = 0$. Taking the limit in (4.2), we conclude

$$u_0(x_0) \geq C_2|x_0 - y_0| \geq C_2 \text{dist}(x_0, \{u_0 = 0\})$$

The other inequality follows immediately from local Lipschitz continuity of u_0.

Since nondegeneracy and Lipschitz continuity holds (5), the uniform density of the set of positivity Ω_0 follows in the same way done in the proof of Corollary (3.5).

Finally, item (6) follows from item (3) together with Theorem 3.8. Indeed, from the convergence in Hausdorff distance, for k large enough, we have:

$$\mathcal{N}_\delta(F(u_0) \cap B_\rho(x_0)) \subset \mathcal{N}_{4\delta}(\partial\Omega^+_{C_1\varepsilon_k}) \cap B_{2\rho}(x_0)$$

We are assuming here, $\varepsilon_k << \delta << \rho << \Delta$. This way, the assumptions of Theorem (3.8) are satisfied, providing the estimate on the Lebesgue measure of the δ-neighborhood. Finally, let us prove the $\mathcal{H}^{N-1}$ Hausdorff measure estimate.

Let $\{B_j\}$ be a covering of $F(u_0) \cap B_\rho(x_0)$ by balls centered in $F(u_0) \cap B_\rho(x_0)$ and radius δ. Clearly

$$\bigcup B_j \subset \mathcal{N}_\delta(F(u_0)) \cap B_{\rho+\delta}(x_0))$$

This way, there exists $\overline{C} = \overline{C}(N)$ such that

$$\begin{aligned}\mathcal{H}^{N-1}_\delta(F(u_0) \cap B_\rho(x_0)) &\leq \overline{C} \sum \text{Area}\left(\partial B_j\right) = \frac{\overline{C}}{\delta} \sum |B_j| \\ &\leq \frac{\overline{C}}{\delta} |\mathcal{N}_\delta(F(u_0)) \cap B_{\rho+\delta}(x_0)| \leq \overline{C}C(\rho+\delta)^{N-1} = \overline{C}C\rho^{N-1} + o(\delta)\end{aligned}$$

Letting $\delta \to 0$, we finish the proof of the Theorem. □

We can also proof the following finer convergence result, which will be employed in the proof of the free boundary condition. This is the content of the next Lemma.

Lemma 4.2 *Let $\Omega' \subset\subset \Omega$. Then $\nabla u_k \to \nabla u_0$ in $L^2_{\text{loc}}(\Omega)$. In particular,*

$$\int_{\Omega'} \langle A(x)\nabla u_k, \nabla u_k\rangle \,\mathrm{d}x \to \int_{\Omega'} \langle A(x)\nabla u_0, \nabla u_0\rangle \,\mathrm{d}x$$

Proof Let $\varphi \in C_0^\infty(\Omega')$, $\varphi \geq 0$ and $\delta > 0$. Since, $Lu_0 = 0$ in Ω_0 and $\eta_0 = (u_0 - \delta)^+\varphi \in H_0^1(\Omega') \cap H_0^1(\Omega_0)$, we have:

$$0 = \int_{\Omega_0} \langle A(x)\nabla u_0, \nabla \eta_0\rangle = \int_{\{u_0>\delta\}\cap\Omega'} \langle A(x)\nabla u_0, \nabla u_0\rangle\, \varphi(x)\mathrm{d}x$$
$$+ \int_{\{u_0>\delta\}\cap\Omega'} u_0 \,\langle A(x)\nabla u_0, \nabla\varphi\rangle \,\mathrm{d}x - \delta \int_{\{u_0>\delta\}\cap\Omega'} u_0\, \langle A(x)\nabla u_0, \nabla\varphi\rangle \,\mathrm{d}x$$

Letting $\delta \to 0$ we find

$$\int_{\Omega'} \langle A(x)\nabla u_0, \nabla u_0\rangle\, \varphi(x)\mathrm{d}x = -\int_{\Omega'} u_0 \,\langle A(x)\nabla u_0, \nabla\varphi\rangle \,\mathrm{d}x \tag{4.3}$$

On the other hand, from the fact that $\beta_{\varepsilon_k} u_k \geq 0$ and u_k is a weak solution, we obtain

$$\int_{\Omega'} \langle A(x)\nabla u_k, \nabla u_k\rangle\, \varphi(x)\mathrm{d}x \leq \int_{\Omega'} -u_k \,\langle A(x)\nabla u_k, \nabla\varphi\rangle \,\mathrm{d}x \tag{4.4}$$

Now, from the local uniform convergence of u_k to u_0 and the weak convergence of ∇u_k to ∇u_0 in $L^2(\Omega')$ and (4.3), (4.4) we conclude,

$$\limsup_{k\to\infty} \int_{\Omega'} \langle A(x)\nabla u_k, \nabla u_k\rangle\, \varphi(x)\mathrm{d}x \leq \int_{\Omega'} \langle A(x)\nabla u_0, \nabla u_0\rangle\, \varphi(x)\mathrm{d}x$$

Since $\nabla u_k \rightharpoonup \nabla u_0$ in $L^2_{loc}(\Omega)$ and H^1 is a uniformly convex space, it is well know in Functional Analysis that, $\nabla u_k \to \nabla u_0$ in $L^2(\Omega')$. This finishes the proof. □

Now, we will prove the the limit function u_0 is a local minimizer of some energy functional. This fact will play a decisive role in obtaining the last measure theoretic properties of the free boundary $F(u_0)$ such as, uniform density of Ω_0 and Ω_0^c as well as the Hausdorff measure totality of the reduced free boundary.

Definition 4.3 *For each $\mathcal{O} \subseteq \Omega$ open, we consider the functional, $E_0\colon H^1(\Omega) \to \mathbb{R}$, given by*

$$E_0(\xi, \mathcal{O}) := \int_{\mathcal{O}} \left\{\frac{1}{2}\langle A(x)\nabla\xi, \nabla\xi\rangle + \Gamma(x)\chi_{\{\xi>0\}}\right\}\mathrm{d}x,$$

and to be compatible with notation presented in Theorem (2.2), we introduce

$$\mathcal{F}_\varepsilon(u, \mathcal{O}) := \int_{\mathcal{O}} \left\{\frac{1}{2}\,\langle A(x)\nabla u, \nabla u\rangle + \Gamma(x)B_\varepsilon(u)\right\}\mathrm{d}x$$

and to simplify notation, we denote $\mathcal{F}_{\varepsilon_k}(u, \mathcal{O}) = \mathcal{F}_k(u, \mathcal{O})$.

As we pointed out earlier, this definiton is motivated by the following,

Theorem 4.4 (Variational Characterization of u_0) *The function* u_0 *is a local minimizer of* E_0 *over* H^1.

Proof Let $B_{r_0} \subset\subset \Omega$ and a function $\xi \in H^1$ with $\xi \equiv u_0$ on ∂B_{r_0}. We need to show that

$$E_0(B_{r_0}, u_0) \le E_0(B_{r_0}, \xi)$$

In order to do that, let us start interpolating u_0 and u_k in a linear fashion, that is, for $h > 0$ small and fixed define

$$\xi_k^h := \begin{cases} u_0 + \dfrac{|x| - r_0}{h}(u_k - u_0) & \text{in } B_{r_0+h} \setminus B_{r_0} \\ \xi & \text{in } B_{r_0}. \end{cases}$$

Therefore,

$$|\nabla \xi_k^h|^2 \le C_0 + 2\frac{|u_k - u_0|^2}{h^2} \quad \text{in } B_{r_0+h} \setminus B_{r_0}.$$

Thus, since $B_{\varepsilon_k} \le \chi_{[0,+\infty)}$

$$\begin{aligned}
\mathcal{J}^k_{r_0+h,h} &:= \int\limits_{B_{r_0+h}} \left\{ \frac{1}{2}\left\langle A(x)\nabla\xi_k^h, \nabla\xi_k^h \right\rangle + B_{\varepsilon_k}(\xi_k^h) \right\} \mathrm{d}x \\
&\le \left(\frac{\Lambda}{2} + 1\right) \int\limits_{B_{r_0+h}\setminus B_{r_0}} \left\{ |\nabla \xi_k^h|^2 + B_{\varepsilon_k}(\xi_k^h) \right\} \mathrm{d}x + \mathcal{F}_k(\xi, B_{r_0}) \\
&\le \left(\frac{\Lambda}{2} + 1\right) \left\{ \overline{C_0}|B_{r_0+h} \setminus B_{r_0}| + \frac{2}{h^2} \int\limits_{B_{r_0+h}\setminus B_{r_0}} |u_k - u_0|^2 \mathrm{d}x \right\} + \mathcal{F}_k(\xi, B_{r_0}) \\
&\le \left(\frac{\Lambda}{2} + 1\right) \left\{ \overline{C_0}|B_{r_0+h} \setminus B_{r_0}| + \frac{2}{h^2} \int\limits_{B_{r_0+h}\setminus B_{r_0}} |u_k - u_0|^2 \mathrm{d}x \right\} + E_0(\xi, B_{r_0}) \\
&\le Cr^{N-1}h + o_k(1) + E_0(\xi, B_{r_0}).
\end{aligned}$$

where o_k means o(1) with respect to k. Now, by Theorem (4.1), applied with $\Omega' = B_{r_0}$, we have for $h >> \varepsilon_k$

$$\begin{aligned}
\int\limits_{B_{r_0}} \Gamma(x)\chi_{\{u_0>0\}}\mathrm{d}x &\le ||\Gamma||_{L^\infty}\big|\Omega_0 \cap B_{r_0}| \le ||\Gamma||_{L^\infty}|\mathcal{N}_h(\Omega_k \cap B_{r_0})| \le ||\Gamma||_{L^\infty} Chr_0{}^{N-1} \\
&\le \overline{C}hr_0{}^{N-1} + \int\limits_{B_{r_0}} \Gamma(x)B_{\varepsilon_k}(u_k)\mathrm{d}x
\end{aligned}$$

and since $u_k \rightharpoonup u_0$ in $H^1(B_{r_0})$,

$$\int\limits_{B_{r_0}} \langle A(x)\nabla u_0, \nabla u_0 \rangle \, \mathrm{d}x \le \liminf_{k\to+\infty} \int\limits_{B_{r_0}} \langle A(x)\nabla u_k, \nabla u_k \rangle \, \mathrm{d}x$$

This implies

$$E_0(u_0, B_{r_0}) \le \liminf_{k\to+\infty} \mathcal{F}_k(u_k, B_{r_0}) + \overline{C}hr_0{}^{N-1}$$

Finally, since u_k is a minimizer of $\mathcal{F}_k$, we have

$$\mathcal{J}^k_{r_0+h,h} \ge \mathcal{J}^k_{r_0,h} \ge \mathcal{F}_k(u_k, B_{r_0})$$

This way,

$$E_0(u_0, B_{r_0}) \le \limsup_{k\to+\infty} \mathcal{J}^k_{r_0+h,h} + \overline{C}hr_0{}^{N-1} \le (C+\overline{C})hr_0{}^{N-1} + E_0(\xi, B_{r_0})$$

Letting $h \to 0$, we conclude the proof. □

Next, we state the density of the zero phase of the solution along the free boundary. The proof is a straightforward modification of Theorem 1.18, in [7].

Theorem 4.5 *Let $x_0 \in F(u_0) \cap \Omega'$. Then there exist universal constants $\tau^\star = \tau^\star(\Omega')$ such that if $\rho \le \frac{\Delta}{4}$*

$$\left|\Omega_0^C \cap B_\rho(x_0)\right| \ge \tau^\star \rho^N$$

To conclude the measure theoretic properties of the free boundary, we state the following

Corollary 4.6 *Assume $x_0 \in F(u_0) \cap \Omega'$. Then the following hold*

(1) *There exist universal constants $\underline{C}, \overline{C}$ depending on Ω' such that*

$$\underline{C}\rho^{N-1} \le \mathcal{H}^{N-1}(F(u_0) \cap B_\rho(x_0)) \le \overline{C}\rho^{N-1} \quad (\rho << \Delta)$$

(2) $\mathcal{H}^{N-1}(F(u_0) \setminus F(u_0)_{\text{red}}) = 0$.

Proof (2) Follows from (1) since $F(u_0)$ coincides with the measure theoretic free boundary $F(u_0)_M$ and $\mathcal{H}^{N-1}(F(u_0)_M \setminus F(u_0)_{\text{red}}) = 0$. Let us prove (1). The second inequality, it is already proven in Theorem (4.1) (6). The first inequality is a consequence of the isoperimetric inequality. Indeed, by Theorem (4.1) (6) that Ω_0 is a set of finite perimeter, we conclude if $\rho << \Delta$

$$\min\left\{\tau, \tau^\star\right\} \rho^{N-1} \le C||D\chi_{\Omega_0}||(B_\rho(x)) \le C\mathcal{H}^{N-1}(F(u_0) \cap B_\rho(x_0))$$

□

5 Free boundary condition

This section is devoted to study the Free boundary condition. The goal is to verify it in a sense which is as classical as possible and discuss the regularity of the free boundary. We prove that condition holds in four (weak) senses and the $C^{1,\gamma}$ regularity of the free boundary if A is Lipschitz continuous. Namely, the conditons are integral sense, measure sense, $\mathcal{H}^{N-1}$ almost everywhere along the reduced free boundary and in the viscosity sense, notion introduced by Caffarelli in the celebrated and by now classical papers [4–6] .

Next, we prove the free boundary condition in a weak sense, that we called Integral sense. Except for the regularity of the free boundary, it is the only result in this paper where we assume that the matrix $A = A(x)$ is Lipschitz continuous. It could be understood as a motivation for what comes next. As usual, Ω' will always denote a compactly contained subset of Ω.

Theorem 5.1 (Free boundary condition – integral sense) *Assume* $A = A(x) \in C^{0,1}(\overline{\Omega})$, $\Gamma \in L^{\infty}(\Omega) \cap H^1_{loc}(\Omega)$. *Let* $B = B_\rho(x_0) \subset\subset \Omega$ *be a ball centered on the free boundary* $F(u_0) = \partial\{u_0 > 0\} \cap \Omega$ *and* $\overrightarrow{\Psi} \in H^1_0(B, \mathbb{R}^N)$. *Then*

$$\lim_{\gamma \to 0^+} \int\limits_{B \cap \{u_0 = \gamma\}} \Big[\langle A\nabla u_0, \nabla u_0\rangle - 2\Gamma\Big] \overrightarrow{\Psi} \cdot \nu \, \mathrm{d}\mathcal{H}^{N-1} = 0, \tag{5.1}$$

where ν *is the outward unit normal to* $\{u_0 > \gamma\}$ *along the surface* $B \cap \{u_0 = \gamma\}$

Proof For the beginning, let us observe that under the assumption of Lipschitz continuity on $A = A(x)$, the minimizers $u_\varepsilon \in C^{1,\alpha}(\overline{\Omega}) \cap H^2_{\mathrm{loc}}(\Omega)$ for any $\alpha \in (0,1)$, by elliptic regularity theory. This way, we can freely use Green's formulas. Let ψ be a function in $H^1_0(B)$. Let us denote by $G = B \cap \{u_0 > 0\}$. In what follows, the index ε should be understood as the sequence $\{\varepsilon_k\}_{k \geq 1}$ given before. If we multiply the equation

$$\mathrm{div}(A\nabla u_\varepsilon) = \Gamma \beta_\varepsilon(u_\varepsilon)$$

by $\psi \partial_k u_\varepsilon$, and integrate it over B, since

$$\mathrm{div}\big(\psi \partial_k u A\nabla u_\varepsilon\big) = \langle A\nabla u_\varepsilon, \nabla(\psi \partial_k u_\varepsilon)\rangle + \psi \partial_k u_\varepsilon \mathrm{div}\big(A\nabla u_\varepsilon\big),$$

we obtain

$$0 = \int\limits_B \{\langle A\nabla u_\varepsilon, \nabla(\psi \partial_k u_\varepsilon)\rangle + \Gamma \beta_\varepsilon(u_\varepsilon)\psi \partial_k u_\varepsilon\} \, \mathrm{d}x. \tag{5.2}$$

Using integration by parts, we obtain

$$-\int\limits_B \langle A\nabla u_\varepsilon, \nabla(\psi \partial_k u_\varepsilon)\rangle \mathrm{d}x = \int\limits_B (\Gamma\psi)\partial_k(B_\varepsilon(u_\varepsilon)) = -\int\limits_B \partial_k(\Gamma\psi) B_\varepsilon(u_\varepsilon)\mathrm{d}x \tag{5.3}$$

but, from the following identity a.e.,

$$\partial_k \langle A\nabla u_\varepsilon, \nabla u_\varepsilon\rangle = 2\langle A\nabla u_\varepsilon, \nabla \partial_k u_\varepsilon\rangle + \langle(\partial_k A)\nabla u_\varepsilon, \nabla u_\varepsilon\rangle$$

We have,

$$\begin{aligned}\langle A\nabla u_\varepsilon, \nabla(\psi \partial_k u_\varepsilon)\rangle &= \langle A\nabla u_\varepsilon, \nabla\psi\rangle \partial_k u_\varepsilon + \langle A\nabla u_\varepsilon, \nabla\partial_k u_\varepsilon\rangle \psi = \langle A\nabla u_\varepsilon, \nabla\psi\rangle \partial_k u_\varepsilon \\ &\quad + \frac{1}{2}\{\partial_k \langle A\nabla u_\varepsilon, \nabla u_\varepsilon\rangle - \langle(\partial_k A)\nabla u_\varepsilon, \nabla u_\varepsilon\rangle\}\psi\end{aligned}$$

From (5.3) and again integrating by parts, we obtain

$$\begin{aligned}&\int\limits_B \left\{\langle A\nabla u_\varepsilon, \nabla\psi\rangle \partial_k u_\varepsilon - \frac{1}{2}\langle A\nabla u_\varepsilon, \nabla u_\varepsilon\rangle \partial_k\psi - \frac{1}{2}\langle(\partial_k A)\nabla u_\varepsilon, \nabla u_\varepsilon\rangle \psi\right\} \mathrm{d}x \\ &\quad = \int\limits_B \partial_k(\Gamma\psi) B_\varepsilon(u_\varepsilon)\mathrm{d}x\end{aligned}$$

By reflexivity, we may assume $B_\varepsilon(u_\varepsilon) \rightharpoonup \phi$ for some function $0 \leq \phi \leq 1$ in $L^2(B)$. But, since $B_\varepsilon(s) = 1$ for $s \geq \varepsilon$ we conclude $\phi \equiv 1$ in G. Now, letting $\varepsilon \to 0$ and taking into account Lemma (4.2), we find

$$\int_B \left\{ \langle A\nabla u_0, \nabla\psi\rangle \partial_k u_0 - \frac{1}{2}\langle A\nabla u_0, \nabla u_0\rangle \partial_k\psi - \frac{1}{2}\langle(\partial_k A)\nabla u_0, \nabla u_0\rangle \psi \right\} \mathrm{d}x = \int_B \partial_k(\Gamma\psi)\phi \mathrm{d}x$$

Since by Theorem (4.1), u_0 is locally Lipschitz and nondegenerate, we are able to reproduce a strip measure estimate like Theorem (3.7) by following ([6], Lemma 10), so

$$|B \cap \{0 < u_0 < \gamma\}| \to 0 \quad \text{as } \gamma \to 0$$

In particular, we can rewrite the equation above in the form

$$\begin{aligned}&\int_{B\cap\{u_0>\gamma\}} \left\{ \langle A\nabla u_0, \nabla\psi\rangle \partial_k u_0 - \frac{1}{2}\langle A\nabla u_0, \nabla u_0\rangle \partial_k\psi - \frac{1}{2}\langle(\partial_k A)\nabla u_0, \nabla u_0\rangle \psi \right\} \mathrm{d}x \\ &\quad +\sigma_1(\gamma) = \int_B \partial_k(\Gamma\psi)\phi \mathrm{d}x \end{aligned} \tag{5.4}$$

with $\lim_{\gamma\to 0} \sigma_1(\gamma) = 0$. By the other hand, since

$$\operatorname{div}(A\nabla u_0)\psi\,\partial_k u_0 = 0 \text{ in } B \cap \{v > \gamma\}$$

we have

$$\begin{aligned}\int_{B\cap\{u_0>\gamma\}} \langle A\nabla u_0, \nabla(\psi\,\partial_k u_0)\rangle \,\mathrm{d}x &= \int_{B\cap\{u_0=\gamma\}} \psi\,\partial_k u_0 \,\langle A\nabla u_0, \nu\rangle \,\mathrm{d}\mathcal{H}^{N-1} \\ &= \int_{B\cap\{u_0=\gamma\}} \langle A\nabla u_0, \nabla u_0\rangle \,\psi\,\nu_k \mathrm{d}\mathcal{H}^{N-1} \end{aligned} \tag{5.5}$$

proceeding now, like we did in (5.3) and taking into account the boundary term that appears in the integration by parts in the second equality, we obtain

$$\begin{aligned}&\int_{B\cap\{u_0>\gamma\}} \langle A\nabla u_0, \nabla(\psi\,\partial_k u_0)\rangle \,\mathrm{d}x \\ &= \int_{B\cap\{u_0>\gamma\}} \langle A\nabla u_0, \nabla\psi\rangle \partial_k u_0 \mathrm{d}x + \frac{1}{2}\int_{B\cap\{u_0>\gamma\}} \partial_k \langle A\nabla u_0, \nabla u_0\rangle \,\psi \mathrm{d}x \\ &\quad - \frac{1}{2}\int_{B\cap\{u_0>\gamma\}} \langle(\partial_k A)\nabla u_0, \nabla u_0\rangle \,\psi \mathrm{d}x \\ &= \int_{B\cap\{u_0>\gamma\}} \langle A\nabla u_0, \nabla\psi\rangle \partial_k u_0 \mathrm{d}x - \frac{1}{2}\int_{B\cap\{u_0>\gamma\}} \langle A\nabla u_0, \nabla u_0\rangle \,\partial_k\psi \mathrm{d}x\end{aligned}$$

$$-\frac{1}{2}\int\limits_{B\cap\{u_0=\gamma\}}\langle A\nabla u_0,\nabla u_0\rangle\,\psi\nu_k\mathrm{d}\mathcal{H}^{N-1}-\frac{1}{2}\int\limits_{B\cap\{u_0>\gamma\}}\langle(\partial_k A)\nabla u_0,\nabla u_0\rangle\,\psi\mathrm{d}x$$

$$=\int\limits_B\partial_k(\Gamma\psi)\phi\mathrm{d}x-\sigma_1(\gamma)+\frac{1}{2}\int\limits_{B\cap\{u_0=\gamma\}}\langle A\nabla u_0,\nabla u_0\rangle\,\psi\nu_k\mathrm{d}\mathcal{H}^{N-1}$$

Now, finally, by (5.5), we arrive at

$$\int\limits_B\partial_k(\Gamma\psi)\phi\mathrm{d}x=\sigma_1(\gamma)+\frac{1}{2}\int\limits_{B\cap\{u_0=\gamma\}}\langle A\nabla u_0,\nabla u_0\rangle\,\psi\nu_k\mathrm{d}\mathcal{H}^{N-1}\tag{5.6}$$

We will finish the proof, if we can show that $\phi\equiv 0$ in $B\setminus G$. Indeed, since $\phi\equiv 1$ in G, we have:

$$\begin{aligned}\sigma_1(\gamma)+\frac{1}{2}\int\limits_{B\cap\{u_0=\gamma\}}\langle A\nabla u_0,\nabla u_0\rangle\,\psi\nu_k\mathrm{d}\mathcal{H}^{N-1}&=\int\limits_B\partial_k(\Gamma\psi)\phi\mathrm{d}x=\int\limits_G\partial_k(\Gamma\psi)\mathrm{d}x\\&=\int\limits_{B\cap\{u_0\geq\gamma\}}\partial_k(\Gamma\psi)\mathrm{d}x+\sigma_2(\gamma)\\&=\int\limits_{B\cap\{u_0=\gamma\}}\Gamma\psi\nu_k\mathrm{d}\mathcal{H}^{N-1}+\sigma_2(\gamma)\end{aligned}$$

with $\lim\limits_{\gamma\to 0}\sigma_2(\gamma)=0$. In particular,

$$\int\limits_{B\cap\{u_0=\gamma\}}(\langle A\nabla u_0,\nabla u_0\rangle-2\Gamma)\,\psi\nu_k\mathrm{d}\mathcal{H}^{N-1}=2(\sigma_2(\gamma)-\sigma_1(\gamma))$$

To Finish, we just need to prove the following,

Claim $\phi\equiv 0$ in $B\setminus G$

In fact, from identity (5.6), taking ψ is supported in the interior of $B\setminus G$, we conclude that

$$\nabla\phi=0\text{ in }\mathcal{D}^{'}(B\setminus G)$$

Thus, ϕ is also constant in $B\setminus G$, say $\phi\equiv\Lambda$ in $B\setminus G$.

We claim $\Lambda=0$. Indeed, let $B_r(\zeta)$ be a ball in the interior $B\setminus G$ and consider $\sigma_0>0$ so small that $B_{r+\sigma}(\zeta)$ is still contained in the interior $B\setminus G$, for all $\sigma\leq\sigma_0$. Let η_σ be a nonnegative smooth cutoff function with the following properties:

- $\eta_\sigma\equiv 0$ in $B_r(\zeta)$.
- $\eta_\sigma\equiv 1$ in $\Omega\setminus B_{r+\sigma}(\zeta)$.

Since u_ε is a global minimizer of the considered variational problem, we have, $\mathcal{F}_\varepsilon(u_\varepsilon)\leq\mathcal{F}_\varepsilon(\eta_\sigma u_\varepsilon)$. Translating this, in integral terms, we find

$$\int\limits_{B_{r+\sigma}(\zeta)}\left\{\frac{1}{2}\langle A\nabla u_\varepsilon,\nabla u_\varepsilon\rangle+\Gamma B_\varepsilon(u_\varepsilon)\right\}\mathrm{d}x\leq\int\limits_{B_{r+\sigma}(\zeta)}\left\{\frac{1}{2}\langle A\nabla\eta_\sigma u_\varepsilon,\nabla\eta_\sigma u_\varepsilon\rangle+\Gamma B_\varepsilon(\eta_\sigma u_\varepsilon)\right\}\mathrm{d}x$$

Thus,

$$\begin{aligned}\int\limits_{B_{r+\sigma}(\zeta)} \Gamma B_\varepsilon(u_\varepsilon)\mathrm{d}x &\le \int\limits_{B_{r+\sigma}(\zeta)} \left\{\frac{1}{2}\langle A\nabla\eta_\sigma u_\varepsilon, \nabla\eta_\sigma u_\varepsilon\rangle + \Gamma B_\varepsilon(\eta_\sigma u_\varepsilon)\right\}\mathrm{d}x \\ &\le \Lambda \int\limits_{B_{r+\sigma}(\zeta)} \left(u_\varepsilon^2|\nabla\eta_\sigma|^2 + \eta_\sigma^2|\nabla u_\varepsilon|^2\right)\mathrm{d}x + ||\Gamma||_{L^\infty}|B_{r+\sigma}(\zeta)\setminus B_r(\zeta)|\end{aligned} \tag{5.7}$$

Since, $u_\varepsilon \to 0$ in $H^1(B_{r+\sigma_0}(\zeta))$, by Lemma 4.2, applying the limit in the inequality above as $\varepsilon \to 0$ we obtain,

$$\int\limits_{B_{r+\sigma}(\zeta)} \Gamma\phi\mathrm{d}x \le ||\Gamma||_{L^\infty}|B_{r+\sigma}(\zeta)\setminus B_r(\zeta)|$$

Letting now, σ goes to zero, we finish the proof. □

We can actually prove a stronger version of the above Theorem by showing that the Free Boundary Condition (5.1) holds for any local minimizer of the functional $E_0(.,\Omega)$. We point out that, any local minimizer of $E_0(.,\Omega)$ is continuous. For the proof, look at Theorem 2.1 in [2].

Although, the following result is proven for Lipschitz Matrices, this will have no effect in treating the free boundary condition for A Holder continuous, since this result will just be employed under circunstances [proof of Theorem (5.8)] where much more regularity is available.

Theorem 5.2 *Assume $A = A(x) \in C^{0,1}(\overline{\Omega})$ and let ξ be a local minimizer of $E_0(.,\Omega)$. Then if $|F(\xi)| = 0$,*

$$\lim_{\gamma\to 0^+} \int\limits_{\partial\{\xi>\gamma\}} \Big[\langle A\nabla\xi, \nabla\xi\rangle - 2\Gamma\Big]\overrightarrow{\Psi}\cdot\nu\, \mathrm{d}\mathcal{H}^{N-1} = 0, \tag{5.8}$$

for any $\overrightarrow{\Psi} \in H_0^1(\Omega,\mathbb{R}^N)$, where ν is the outward unit normal to $\{\xi > \gamma\}$ along the surface $\partial\{\xi > \gamma\}$

Proof For $|\gamma|$ small enough, let us define $\tau_\gamma : \Omega \to \Omega$ defined by $\tau_\gamma(x) = x + \gamma\overrightarrow{\Psi}(x)$. Denoting $A_\gamma = A\circ\tau_\gamma$ and setting $\xi_\gamma = \xi\circ(\tau_\gamma)^{(-1)}$ since it has the same boundary values of ξ,

$$\begin{aligned} 0 &\le E_0(\xi_\gamma,\Omega) - E_0(\xi,\Omega) \\ &= \int\limits_{\{\xi>0\}} \left[\left\{\frac{1}{2}\left\langle A_\gamma(x)\nabla\xi(D\tau_\gamma)^{(-1)}, \nabla\xi(D\tau_\gamma)^{(-1)}\right\rangle + \Gamma\circ\tau_\gamma\right\}\det D\tau_\gamma - \frac{1}{2}\langle A\nabla\xi,\xi\rangle - \Gamma\right]\mathrm{d}x \\ &= \gamma\int\limits_{\{\xi>0\}} \left\{\frac{1}{2}\langle A\nabla\xi,\nabla\xi\rangle + \Gamma\right\}\mathrm{div}(\overrightarrow{\Psi})\mathrm{d}x \\ &\quad + \gamma\int\limits_{\{\xi>0\}} \left\{\frac{1}{2}\left\langle (DA)\overrightarrow{\Psi}\nabla\xi, \nabla\xi\right\rangle - \left\langle A\nabla\xi, D\overrightarrow{\Psi}\nabla\xi\right\rangle + \left\langle \nabla\Gamma, \overrightarrow{\Psi}\right\rangle\right\}\mathrm{d}x + o(\gamma)\end{aligned}$$

We can conclude from that the linear term in γ vanishes. It is easy to check that $L\xi = 0$ in $\{\xi > 0\}$. To see this, we just use that $E_0(\xi + t\eta,\Omega) \ge E_0(\xi,\Omega)$ with

$\eta \in C_0^\infty(\{\xi > 0\})$, $|t|$ small enough. This way,

$$\begin{aligned} 0 &= \int\limits_{\{\xi>0\}} \operatorname{div}\left\{\left(\frac{1}{2}\langle A\nabla\xi, \nabla\xi\rangle + \Gamma\right)\overrightarrow{\Psi} - \left\langle\overrightarrow{\Psi}, \nabla\xi\right\rangle A\nabla\xi\right\} \mathrm{d}x \\ &= \lim_{\gamma\to 0} \int\limits_{\partial\{\xi>\gamma\}} \left\langle\left\{\left(\frac{1}{2}\langle A\nabla\xi, \nabla\xi\rangle + \Gamma\right)\overrightarrow{\Psi} - \left\langle\overrightarrow{\Psi}, \nabla\xi\right\rangle A\nabla\xi\right\}, \nu\right\rangle \mathrm{d}\mathcal{H}^{N-1} \\ &= \lim_{\gamma\to 0} \int\limits_{\partial\{\xi>\gamma\}} (\Gamma - \frac{1}{2}\langle A\xi, \xi\rangle)\left\langle\overrightarrow{\Psi}, \nu\right\rangle \mathrm{d}\mathcal{H}^{N-1} \end{aligned}$$

□

Remark 5.3 We point out that the assumption that the free boundary has measure zero in the above Theorem is really not necessary, since, this can actually be proven for any local minimizer following the procedure developed in [1].

Remark 5.4 Theorem (5.8) shows in particular that if the free boundary $\partial\{\xi > 0\}$ is C^1 and ξ is C^1 in $\{\xi > 0\}$ uniformly up to the free boundary, then

$$\langle A\nabla\xi, \nabla\xi\rangle = 2\Gamma \text{ on the free boundary.}$$

We state now our next result. The proof is based on Green's representation formula an Harnack inequality which is also available for operators in divergence form. It follows by a straightforward modification of Theorem (1.19) in [7].

Theorem 5.5 *There exist constants $\underline{C}, \overline{C}$ depending on Ω', such that*

$$\underline{C}\rho^{N-1} \leq Lu_0(B_\rho) \leq \overline{C}\rho^{N-1} \tag{5.9}$$

for any (small) ball $B_\rho = B_\rho(x_0) \subset \Omega', x_0 \in F(u_0) = \partial\{u_0 > 0\} \cap \Omega$.

Before we continue, we sate the main properties of blow-up sequences.

Definition 5.6 *Let ξ be a local minimum of $E_0(.,\Omega)$, and let $B_{\rho_k}(x_k)$ be a sequence of balls in Ω with $\rho_k \to 0, x_k \to x_0 \in \Omega$ with $\xi(x_k) = 0$. The sequence of functions*

$$\xi_k(x) = \frac{1}{\rho_k}\xi(x_k + \rho_k x)$$

is called a blow-up sequence.

Since $|\nabla\xi_k(x)| \leq C$ in every compact set of $\mathbb{R}^N$, if k is large enough, and since $\xi_k(0) = 0$, it follows that for a subsequence,

$$\begin{aligned} \xi_k &\to \xi_\infty \text{ in } C^\alpha_{loc} \quad \forall\alpha, 0 < \alpha < 1 \\ \nabla\xi_k &\to \nabla\xi_\infty \quad \text{weakly star in } L^\infty_{\mathrm{loc}} \end{aligned}$$

The function ξ_∞ is called a blow-up limit.

Below, we sumarize the main properties of the blow-up limit. A direct proof is given in ([10], Lemma 3.6, pp. 281) and ([1], pp. 120).

Proposition 5.7 (blow-up limit properties) *The following properties hold*

(1) $\partial\{\xi_k > 0\} \to \partial\{\xi_\infty > 0\}$ *in the Hausdorff distance;*
(2) $\chi_{\{\xi_k>0\}} \to \chi_{\{\xi_\infty>0\}}$ *in L^1_{loc};*

(3) *If* $x_k \in \partial\{\xi_k > 0\}$ *then* $0 \in \partial\{\xi_\infty > 0\}$*;*

(4) $\nabla\xi_k \to \nabla\xi_\infty$ *a.e;*

(5) *If* Γ *is continuous, then* ξ_∞ *is an absolute minimizer in every bounded domain for the following functional*

$$E_{x_0}(\xi, \Omega^{'}) = \int_{\Omega^{'}} \left\{\langle A(x_0)\nabla\xi, \nabla\xi\rangle + \Gamma(x_0)\chi_{\{\xi>0\}}\right\} \mathrm{d}x$$

Now, we are ready to prove the free boundary condition in the measure sense. This is the content of the following Theorem. Related results can be found in [1] and [8].

Theorem 5.8 (Free boundary condition – measure sense) *Assume* Γ *is continuous. There exists a* $\mathcal{H}^{N-1}$ *measurable function g such that*

(1) $Lu_0 = g\mathcal{H}^{N-1}\lfloor\partial\{u_0 > 0\}$, *i.e,*

$$-\int_{\Omega} \langle A(x)\nabla u_0, \nabla\varphi\rangle\, \mathrm{d}x = \int_{\Omega\cap\{u_0>0\}} \varphi g \mathrm{d}\mathcal{H}^{N-1} \quad \forall\varphi \in C_0^\infty(\Omega)$$

(2) *There exists universal constants* $\underline{C}, \overline{C}$ *depending on* $\Omega^{'}$ *such that*

$$\underline{C} \le g(x) \le \overline{C} \qquad \mathcal{H}^{N-1}\ a.e\ x \in \Omega^{'} \cap \{u_0 > 0\}$$

(3) *If* $x_0 \in F(u_0)_{red}$ *with* $\Theta^{\star n-1}(\mathcal{H}^{N-1}\lfloor F(u_0), x_0) \le 1$ *and* ν *is the outward unit normal vector to* Ω_0 *at* x_0 *then,*

$$g(x_0) = \sqrt{2\Gamma(x_0)\,\langle A(x_0)\nu, \nu\rangle}$$

Proof Let $g = \mathrm{d}Lu_0/\mathrm{d}\mathcal{H}^{N-1}$, the Radon–Nikodym derivative. Let us observe now that Theorem (5.5) says that $\mu \ll \mathcal{H}^{N-1}$. By Lebesgue Decomposition Theorem for Radon measures and estimate (5.9) we conclude (1) and (2). Let us concentrate on (3) now. Let $x_0 \in F(u_0)_{\mathrm{red}}$. Let us assume without lost of generality that $\nu(x_0) = e_N$. A well know property of sets of finite perimeter says that the blow-up limits at x_0 of the sets Ω_0 and Ω_0^c are, respectively, the half planes

$$H_{x_0}^- = \{x\,; (x_0)_N - x_N < 0\} \text{ and } H_{x_0}^+ = \{x;\ (x_0)_N - x_N > 0\}$$

(for a proof of this fact, look at ([9], Theorem 1, pp 199–203))

By the other hand, from proposition (5.7),the blow-up of the sets Ω_0 and Ω_0^c are given as well and respectively, by

$$x_0 + \{u_\infty < 0\} \text{ and } x_0 + \{u_\infty > 0\}$$

Thus, if u_∞ is a blow-up limit with respect to x_0 of u_0, using the fact that $\Theta^{\star n-1}(\mathcal{H}^{N-1}\lfloor F(u_0), x_0) \le 1$ we can follow the proof of Theorem 3.5 in [2] and Theorem 4.8 in [1] to conclude from the analysis above that $u_\infty > 0$ in $H^- = \{x;\ x_N < 0\}$, $u_\infty = 0$ in $H^+ = \{x;\ x_N \ge 0\}$ and

$$L_{A(x_0)}(u_\infty) = 0 \quad \text{in } \{x_N < 0\}$$

Reflecting u_∞ in an odd fashion ([7], remark 11.1), we obtain $\widetilde{u}_\infty$ satisfying,

$$L_{\widetilde{A}(x_0)}(\widetilde{u}_\infty) = 0 \quad \text{in } \mathbb{R}^N$$

Where $\widetilde{A}(x_0)$ is a constant matrix. Since u_∞ is globally Lipschitz, it follows by Liouville theorem that $\widetilde{u}_\infty$ is a linear function and therefore $u_\infty(x) = \theta x_n^-$, for some positive θ. By proposition (5.7)5), u_∞ is an absolute minimizer of E_{x_0} and since it has smooth free boundary, expression (5.8) implies that

$$\theta = \sqrt{\frac{2\Gamma(x_0)}{\langle A(x_0)e_N, e_N\rangle}} \tag{5.10}$$

In particular, u_0 has a unique blow-up limit at x_0, namely

$$u_\infty(x) = \sqrt{\frac{2\Gamma(x_0)}{\langle A(x_0)\nu, \nu\rangle}}\, \langle x, \nu\rangle^- \tag{5.11}$$

By the other hand, if we proceed by repeating the arguments of ([1], Theorem 4.8, pp. 120–121), we obtain

$$-\int\limits_{B_r\cap\{x_N<0\}} \langle A(x_0)\nabla u_\infty, \nabla\varphi\rangle \,\mathrm{d}x = g(x_0) \int\limits_{B_r\cap\mathbf{R}^{N-1}} \varphi(x^{'}, 0)\mathrm{d}\mathcal{H}^{N-1}$$

for every $\varphi \in C_0^1(B_r)$. Since $L_{A(x_0)}(u_\infty) = 0$ in $\{x_N < 0\}$, from the boundary regularity, we conclude that the condition above is actually satisfied in the classical sense, i.e,

$$g(x_0) = -\langle A(x_0)\nabla u_\infty, e_N\rangle \text{ on } \{x_N = 0\}$$

Using now the expression for u_∞ and θ found above we conclude,

$$\theta = \frac{g(x_0)}{\langle A(x_0)e_N, e_N\rangle} \tag{5.12}$$

Comparing (5.10) and (5.12), we conclude the Theorem. □

We can now state the following corollary which provides the free boundary condition in the pointwise sense.

Corollary 5.9 (*Free boundary condition – pointwise sense*) *For $\mathcal{H}^{N-1}$ a.e. $x_0 \in F(u_0)$, the following asymptotic development holds*

$$u_0(x) = \sqrt{\frac{2\Gamma(x_0)}{\langle A(x_0)\nu, \nu\rangle}}\, \langle x - x_0, \nu\rangle^+ + o(|x - x_0|)$$

where ν is the inward unit normal vector to $\{u_0 > 0\}$ at x_0.

Proof Indeed, let $x_0 \in F(u_0)_{\text{red}}$ such that $\Theta^{\star n-1}(\mathcal{H}^{N-1}\lfloor F(u_0), x_0) \leq 1$. As a consequence of the uniqueness of the blow-up limit with respect to such a point (fact obtained in the proof of previous result) and the characterization given by (5.11), we have

$$u_\tau(x - x_0) = \frac{u(\tau(x - x_0))}{\tau} \to \theta\, \langle x - x_0, \nu\rangle^+ \text{ as } \tau \to 0$$

where θ is given by (5.10). Since $\mathcal{H}^{N-1}(F(u_0) \setminus F(u_0)_{\text{red}}) = 0$ and the upper density $\Theta^{\star n-1}(\mathcal{H}^{N-1}\lfloor F(u_0), x_0) \leq 1$ for $\mathcal{H}^{N-1}$ a.e. on $F(u_0)$ the proof is complete. □

We now move towards the free boundary condition in the viscosity sense. This notion was introduced by Luis Caffarelli in [4–6]. First, we need to state a result. For the proof, look [5] and [7].

Proposition 5.10 (*Linear behavior at regular boundary points*) *Assume u be a nonnegative Lipschitz L-Harmonic function in a domain Ω such that u vanishes in $B_r(x_0) \cap \partial\Omega$ for some $x_0 \in \partial\Omega$ and $r > 0$.*

(1) *If there exists $B = B_\rho(y_0) \subset \Omega$ such that $B \cap \partial\Omega = \{x_0\}$. Then, the following development holds in B*

$$u(x) = \theta \left\langle x - x_0, \nu \right\rangle^+ + o(|x - x_0|)$$

with $\theta > 0$, where ν is the unit normal to ∂B inward to Ω.

(2) *If there exists $B = B_\rho(y_0) \subset \Omega^c$ such that $B \cap \partial\Omega = \{x_0\}$, then extending u by zero outside Ω, the following development holds*

$$u(x) = \theta \left\langle x - x_0, \nu \right\rangle^+ + o(|x - x_0|)$$

with $\theta \geq 0$, where ν is the unit normal to ∂B inward to Ω. Moreover, if $\theta > 0$, then B is tangent to $\partial\Omega$ at x_0.

We recall that a point $x_0 \in F(u_0)$ is said to be regular if it has a one-sided ball, i.e, there exists $B_\rho(x_0)$, $x_0 \in \partial B_\rho(x_0)$ and $B_\rho(x_0)$ is contained either in Ω_0 or $(\Omega \setminus \Omega_0)^\circ$.

Theorem 5.11 (*Free boundary condition in the viscosity sense*) *Assume that Γ is continuous. If $x_0 \in F(u_0)$ is a regular point and $B = B_\rho(y_0)$ is the correpondent touching ball, then*

$$u_0(x) = \sqrt{\frac{2\Gamma(x_0)}{\left\langle A(x_0)\nu, \nu \right\rangle}} \left\langle x - x_0, \nu \right\rangle^+ + o(|x - x_0|)$$

where ν is the unit normal to $\partial B_\rho(y_0)$ pointing inward to Ω_0. That is,

$$u_\nu^2 = \frac{2\Gamma}{\left\langle A\nu, \nu \right\rangle} \qquad \text{along } F(u_0)$$

holds in the Caffarelli's viscosity sense.

Proof We can proceed exactly as in [7], pp.23. In any case where B_ρ is placed at, by proposition (5.10), nondegeneracy, uniform density estimate of Ω_0^c (Theorem 4.5) and the monotonicity formula ([6], Lemma 1), we conclude that

$$u_0(x) = \theta_0 \left\langle x - x_0, \nu \right\rangle^+ + o(|x - x_0|), \text{ with } \theta_0 \geq 0$$

This implies in particular that any blow up sequence $\dfrac{u(x_0 + \rho_k x)}{\rho_k}$ with $\rho_k \to 0$ [as in definiton (5.6)] converges uniform in compact subsets to

$$u_\infty(x) = \theta_0 \left\langle x, \nu \right\rangle^+$$

Now, by proposition (5.7), u_∞ is an absolute minimizer of E_{x_0} and since it has smooth free boundary, by expression (5.8), we conclude that

$$\theta = \sqrt{\frac{2\Gamma(x_0)}{\langle A(x_0)\nu, \nu\rangle}}$$

Concluding the proof of the Theorem. □

Finally, we address the free boundary regularity. Let us state the following

Theorem 5.12 ([11] – Theorem 1.3) *Assume $A = A(x)$ is Lipschitz continuous and u is a viscosity solution of the following free boundary problem*

$$\begin{aligned} Lu = div(A(x)\nabla u) = 0 \ in \ \Omega^+ = \Omega \cap \{u > 0\} \ and \ (\Omega \setminus \Omega^+)^\circ \\ u_\nu^+ = G(u_\nu^-, \nu, x) \end{aligned}$$

where $\Omega = B_1^{'}(0) \times (-1, 1), B_1^{'}(0)$ unit ball in $\mathbb{R}^{N-1}$ and

- (i) *There exist positive numbers α_0, α_1 such that $\alpha_0 \leq \frac{u^+(x)}{d(x, F(u))} \leq \alpha_1$*
- (ii) *$G(0, \nu, x) > C_0 > 0, G(z, \cdot, \cdot)$ Lipschitz continuous, strictly increasing in z and for some large constant $N > 0$, independent of ν and x, $z^{-N}G(z, \cdot, \cdot)$ is decreasing in $(0, +\infty)$.*
- (iii) *There exist $\overline{\theta} < \frac{\pi}{2}$ and $\overline{\varepsilon} > 0$ such that for some $\varepsilon, 0 < \varepsilon < \overline{\varepsilon}, F(u)$ is contained in a ε-neighborhood of the graph of a Lipschitz function $x_n = g(x^{'})$ with Lipschitz norm*

$$Lip(g) \leq \tan\left(\frac{\pi}{2} - \overline{\theta}\right)$$

then, in $B_{1/2}^{'}(0), g$ is a $C^{1,\gamma}$ function where γ is a universal constant.

Theorem 5.13 *Assume that $A = A(x), \Gamma = \Gamma(x) \in C^{0,1}(\overline{\Omega})$ and let u_0 be the limit function considered in Theorem (4.1). Then, the free boundary $F(u_0) = \partial\{u_0 > 0\} \cap \Omega$ is a $C^{1,\gamma}$ surface in a neighborhood of $\mathcal{H}^{n-1}$ a.e. point $x_0 \in F(u_0)_{\text{red}}$. In particular, $F(u_0)$ is a $C^{1,\gamma}$ surface in a neighborhood of $\mathcal{H}^{N-1}$ a.e. point in $F(u_0)$.*

Proof Let us just observe that u_0 is a viscosity solution of and admissible free boundary problem. Here,

$$G(z, \nu, x) = \sqrt{z + \frac{2\Gamma(x)}{\langle A\nu, \nu\rangle}}$$

Furthermore, it is locally Lipschitz and it has linear growth away from the free boundary. Furthermore, since $F(u_0)_{\text{red}}$ has full $\mathcal{H}^{N-1}$ measure in $F(u_0), u_0$ is for $\mathcal{H}^{N-1}$ a.e. point on $F(u_0)$ a 1-plane solution. In particular, in any such point x_0, a suitable dilation

$$(u_0)_\tau(x - x_0) = \frac{u(\tau(x - x_0))}{\tau}, \quad \tau \text{ small},$$

falls under conditons of Theorem (1.3) in [11], concluding the Theorem. □

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