Vector Calculus - Extra Exercises

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1 Problems

<u>Remark:</u> In problems $1)$ and $2)$ assume that F, G are smooth vector fields in $\mathbb{R}^2$and φ is a smooth function.

Problem 1) Show that the following relations hold

a) $curl(\nabla f) = 0$

b) $div(curl(F)) = 0$

c) $div(\varphi F) = \varphi div(F) + \nabla\varphi \cdot F$

Problem 2) Show that the following relations hold

a) $curl(\varphi F) = \varphi curl(F) + \nabla\varphi \times F$

b) $div(F \times G) = curl(F) \cdot G - F \cdot curl(G)$

c) Conclude from $b)$ that the cross product of two irrotational vector fields is a divergence free vector field.

<u>Remark:</u> For questions $3)$ and $4)$ discussion on pages 980 and 981 of the book may be helpful.

Problem 3) Suppose an object moves under the influence of a smooth force along a smooth curve parametrized by

$$C: \quad r(t) = (x(t), y(t), z(t)), \quad a \leq t \leq b$$

Use the second Newton's law $(F(r(t)) = ma(t) = mr''(t))$ to show that the work done by the force F is the variation of the Kinect energy at the end points of the curve.

$$W = \int_C F \cdot dr = \mathcal{K}(b) - \mathcal{K}(a), \qquad \text{where Kinect energy is given by } \mathcal{K}(t) = \frac{m}{2}||r'(t)||^2$$

Problem 4) Suppose a particle of mass m is moving under the a conservative smooth vector field $F = -\nabla\varphi$. In other words,

$$mr''(t) = F(r(t)) = -\nabla\varphi(r(t))$$

Show the Law of the conservation of the Energy of Newton Mechanics that says

$$\text{Kinect Energy (t) + Potential Energy (t)} = \mathcal{K}(t) + \varphi(r(t)) = \text{ CONSTANT}$$

Problem 5) Let the curve C be a smooth parametrized by

$$r(t) = (\cos(\alpha(t)), \sin(\alpha(t))) \quad \text{for} \quad a \leq t \leq b$$

where $\alpha : [a, b] \to \mathbb{R}$ is a smooth function.

Show that

a) Show that

$$\int_C \frac{-ydx}{x^2+y^2} + \frac{xdy}{x^2+y^2} = \alpha(b) - \alpha(a)$$

b) If C is a closed curve, conclude that

$$\int_C \frac{-ydx}{x^2+y^2} + \frac{xdy}{x^2+y^2}$$

is an integer number.

Problem 6) If C is a closed curve, the number

$$\eta(C) := \frac{1}{2\pi}\int_C \frac{-ydx}{x^2+y^2} + \frac{xdy}{x^2+y^2}$$

is always an integer called the widing number of C. It represents the number of "turns" the curves does around the origin.

a) Compute the winding number of the curve C parametrized by

$$r(t) = (\cos(kt), \sin(kt)), 0 \leq t \leq 2\pi)$$

b) Compute the $curl(F)$ where

$$F(x, y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$$

c) Is F a conservative vector field? Explain.

Remark: In what follows, we will be referring to the Divergence Theorem which is summarized in the following formula

$$\int\int_D div(F)dA = \int_C F\cdot\nu ds$$

Problem 7) Prove the first Green's identity

$$\int\int_D \big(f\Delta g dA + \nabla f\cdot\nabla g\big)dA = \int_C f\frac{\partial g}{\partial\nu}ds$$

where f, g are twice continuously differentiable functions in the region $\overline{D}$, ν is the exterior unit normal vector along C and $\frac{\partial g}{\partial\nu} = \nabla g\cdot\nu$ is the exterior normal dervivative on the boundary curve C and $\Delta g = g_{xx} + g_{yy} = div(\nabla g)$. (Hint: Apply divergence theorem with $F = f\nabla g$ and use problem $1)c)$).

Problem 8)

a) If g is smooth in the region $\overline{D}$ then

$$\int\int_D \Delta g dA = \int_C \frac{\partial g}{\partial\nu}ds$$

Hint: Use the first Green's identity in the previous problem

b) Conclude that if g is harmonic, i.e, $\Delta g = 0$, then

$$\int_C \frac{\partial g}{\partial\nu}ds = 0$$

Problem 9) Under the same conditions of problem 5, prove the second Green's identity

$$\int\int_D (f\Delta g - g\Delta f)dA = \int_C (f\frac{\partial g}{\partial\nu} - g\frac{\partial f}{\partial\nu})ds$$

Problem 10) Let g be a continuous function of one variable and let

$$F(x,y,z) = g(|x|^2 + |y|^2 + |z|^2)(x,y,z)$$

a) Show that F is conservative (Hint: try $\varphi(x,y,z) = \frac{1}{2}h(x^2+y^2+z^2)$ for a convenient choice of h)

b) Show that F is irrotational.

OPTIONAL PROBLEM - (5 extra points)

No partial credit. For the solution to be considered correct it has to be perfect. Furthermore, if the solution is correct (perfect) in order to get the (5 points) student has to present/explain in a flawless way the solution to me in my office in an arranged time.

Definition 1.1. We say that a function g is radial if $g(x,y,z) = h(\sqrt{x^2+y^2+z^2})$.

Definition 1.2. We say that F is a central vector field if F is of the following form: $F(x,y,z) = h(x,y,z)(x,y,z)$ for some smooth function h.

Prove the following Theorem:

Theorem 1.3. *Let F be a conservative smooth vector field with potential V, i.e,*

$$F(x,y,z) = \nabla V(x,y,z)$$

Prove that F is a central vector field if and only if its potential V is radial. //
Furthermore, show that in this case there is a smooth function φ such that

$$F(x,y,z) = \varphi(\sqrt{x^2+y^2+z^2})(x,y,z)$$