Increasing bounded sequences converge

Theorem 11.1.10 says that increasing bounded sequences converge. The text does not have a proof of this theorem, so I will give you one here.

Theorem 9.6. *If $\{a_n\}_{n+1}^{\infty}$ is a bounded increasing sequence (that is, there is a constant M where $|a_n| \leq M$ for all n, and $a_n < a_{n+1}$ for all n), then*

$$\lim_{n\to\infty} a_n$$

exists and is a finite number.

The basic idea of this proof is to look at decimal expansions of the a_n's. For example, here is a list of decimal expansions to about 15 digits of a sequence generated by $a_{n+1} = \cos(\cos a_n)$ with $a_1 = 0$. This is an increasing sequence.

n	a_n
1	0.00000000000000
2	0.54030230586814
3	0.65428979049778
4	0.70136877362276
5	0.72210242502671
6	0.73140404242251
7	0.73560474043635
8	0.73750689051324
9	0.73836920412232
0	0.73876031987421
11	0.73893775671534
12	0.73901826242741
13	0.73905479074692
15	0.73907136529894
16	0.73907888599499
17	0.73908229852240
18	0.73908384696500
19	0.73908454957521
20	0.73908486838671

If we look down the first column of digits after "0.", we see that the first digit is increasing from 0 to 5 to 6 to 7, and then stays at 7. (The first digit can't decrease because there are no prior digits, and a_n is an increasing sequence.) Once the first digit has become 7, the second digit increases from 0 to 2 to 3, and then seems to be stuck at 3. Further inspection seems to indicate that once this happens the third digit is increasing and soon settles down to 9. So it seems that the limit is something like 0.739.... Now we make this more formal.

Proof. We start with a bounded, increasing sequence $\{a_n\}_{n=m}^{\infty}$. Let's suppose that the bound is M so that $|a_n| \leq M$ for all $n \geq m$. Put $b_n = (a_n + M)/(2M)$. Then $0 \leq b_n < 1$ for all n, and $b_{n+1} > b_n$ for all n. If $\{b_n\}_{n=m}^{\infty}$ is convergent and has a limit L, then $\{a_n\}_{n=m}^{\infty}$ is also convergent with limit $2ML - M$. so we just need to show that $\{b_n\}_{n=m}^{\infty}$ has a limit.

Write out each number b_n in terms of its decimal expansion:

$$\begin{aligned} b_1 &= 0.b_{1,1}b_{1,2}b_{1,3}b_{1,4}\dots \\ b_2 &= 0.b_{2,1}b_{2,2}b_{2,3}b_{2,4}\dots \\ b_3 &= 0.b_{3,1}b_{3,2}b_{3,3}b_{3,4}\dots \\ &\vdots \end{aligned}$$

Each $b_{n,k}$, being the kth digit of b_n, is one of 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9. Since the sequence b_n is increasing, so too are the first digits $b_{1,1}$, $b_{2,1}$, $b_{3,1}$, etc. Since there are only finitely many possibilities, and these digits are increasing, there can only be a finite number of changes (nine at most in fact: $0 \to 1 \to 2 \to 3 \cdots \to 9$). So after a finite number of entries the first digit becomes constant; call this constant digit b_1^*. Let N_1 be the entry when the first digit becomes b_1^*.

If we now look at the column of second digits after row N_1 in our list, we see that these too must also be increasing since the first digit is now constant, and the sequence $\{b_n\}$ is increasing. Again, there can only be a finite number changes after row N_1, so the second digit must also eventually become constant b_2^*. Let N_2 be the row where this first happens.

The same thing happens for the third row: the third digit must eventually become constant, call it b_3^*, and call the row where this first happens N_3. We can continue in this fashion showing that *all* the digits eventually become constant. For the kth digit, let b_k^* be this eventual constant digit, and suppose that it happens first in row N_k.

I say that the number $L = 0.b_1^*b_2^*b_3^*\dots$ (in its decimal expansion) is the limit of $\{b_n\}_{n=m}^{\infty}$.

Suppose I am challenged with a tolerance $\epsilon > 0$. Now I can find $k >$

$\log_{10}(1/\epsilon)$, and put $N = N_k$. If $n \geq N = N_k$, then

$$\begin{aligned} |b_n - L| &= |0.b_{n,1}b_{n,2}\ldots b_{n,k}b_{n,k+1}\ldots - 0.b_1^*b_2^*\ldots b_k^*b_{k+1}^*\ldots| \\ &= |0.b_1^*b_2^*\ldots b_k^*b_{n,k+1}\ldots - 0.b_1^*b_2^*\ldots b_k^*b_{k+1}^*\ldots| \\ &= |0.00\ldots 0b_{n,k+1}\ldots - 0.00\ldots 0b_{k+1}^*\ldots| \leq 10^{-k} < \epsilon \end{aligned}$$

since $b_{n,j} = b_j^*$ for $j = 1, 2, \ldots, k$. So L really is the limit of $\{b_n\}_{n=m}^{\infty}$, and so both $\{b_n\}_{n=m}^{\infty}$ and $\{a_n\}_{n=m}^{\infty}$ converge, as we wanted to show. △

This theorem can be used in a wide range of situations where you need to show a sequence converges, but you don't know what the limit is. Other theorems based on this one include the Alternating Series Test (section 11.5) which says that if $\{a_n\}_{n=m}^{\infty}$ is a decreasing sequence of positive numbers with limit zero, then $\sum_{n=m}^{\infty}(-1)^n a_n$ converges; another consequence is the Absolute Convergence Theorem (section 11.6) which says that if $\sum_{n=m}^{\infty}|a_n|$ converges, then so does $\sum_{n=m}^{\infty} a_n$.

Heine-Borel Theorem

Another useful theorem to know about is the Heine–Borel Theorem. This is not listed in Stewart, but is interesting in itself.

Theorem (Heine–Borel). *If $\{a_n\}_{n+1}^{\infty}$ is a bounded sequence, then there is a convergent subsequence. That is, there is a sequence $\{n_i\}_{i=1}^{\infty}$ with $n_{i+1} > n_i$ for all i, where $\{a_{n_i}\}_{i=1}^{\infty}$ is a convergent sequence.*

The proof of this theorem is similar to the proof of the convergence of increasing bounded sequences.

Proof. Suppose that each a_n is between zero and one. Write out each number a_n in the sequence in terms of its decimal expansion: $a_n = 0.a_{n,1}a_{n,2}a_{n,3}\ldots$. If we write the decimal expansions one above the other we get a table like this:

$$\begin{aligned} a_1 &= 0.a_{1,1}a_{1,2}a_{1,3}a_{1,4}\ldots \\ a_2 &= 0.a_{2,1}a_{2,2}a_{2,3}a_{2,4}\ldots \\ a_3 &= 0.a_{3,1}a_{3,2}a_{3,3}a_{3,4}\ldots \\ &\vdots \end{aligned}$$

Since there are only a finite number of digits, looking down the digits in the first column after the decimal point ($a_{1,1}$, $a_{2,1}$, $a_{3,1}$, etc.), there must be at least one digit that occurs infinitely often. Pick (say) the smallest of these digits, and call it a_1^*, and make n_1 the first row n where $a_{n,1} = a_1^*$.

Now look down the second column of digits after the decimal point, but only at the rows where the first digit is a_1^*. Again there must be some digit that is repeated infinitely many times; pick (say) the smallest of these and call it a_2^*, and make n_2 the first row $n > n_1$ where $a_{n,1} = a_1^*$ and $a_{n,2} = a_2^*$.

We can repeat this process for the third column of digits after the decimal point; in the infinitely many rows where $a_{n,1} = a_1^*$, $a_{n,2} = a_2^*$ there must be a digit that is repeated infinitely many times. Pick (say) the smallest of these and call it a_3^*. Make n_3 the first row with $n > n_2$ where $a_{n,1} = a_1^*$, $a_{n,2} = a_2^*$, and $a_{n,3} = a_3^*$.

Continuing in this way we can find a sequence $n_1 < n_2 < n_3 < \cdots$ where the first k digits of a_{n_k} are $0.a_1^*a_2^*a_3^*\ldots a_k^*$. I claim that $\{a_{n_k}\}_{k=1}^{\infty}$ is a convergent sequence and its limit is $L = 0.a_1^*a_2^*a_3^*\ldots$. For any $\epsilon > 0$, put $K > \log_{10}(1/\epsilon)$, a positive whole number. Then for $k \geq K$,

$$\begin{aligned} |a_{n_k} - L| &= |0.a_1^*a_2^*\ldots a_k^*a_{n_k,k+1}\ldots - 0.a_1^*a_2^*\ldots a_k^*a_{k+1}^*\ldots| \\ &= |0.00\ldots 0a_{n_k,k+1}\ldots - 0.00\ldots 0a_{k+1}^*\ldots| \leq 10^{-k} < \epsilon. \end{aligned}$$

Thus $\lim_{k\to\infty} a_{n_k} = L$. △

The Heine–Borel Theorem is used in many situations in basic analysis.