Questions for a second practice exam

June 10, 2005

This is a three hour closed book exam. Work four questions from part1 and four questions from part 2.

1 Part 1: General Topology

1. Suppose that X and Y are connected nonempty topological spaces. Prove that $X \times Y$ is connected.

2. Suppose that for each $\alpha \in A$ the space X_α is regular. Prove that $\prod_{\alpha \in A} X_\alpha$ is regular.

3. Suppose that X is compact and let $\pi : X \times Y \to Y$ be projection onto the second factor. Prove that π is closed.

4. Suppose that $\{U_i\}$, $i \in \{1, \ldots n\}$ is a finite open cover of the compact Hausdorff space X. Prove that there is an open cover V_j with the property that for each j there is an i so that $\overline{V}_j \subset U_i$ for all i. We are using the overline to denote closure.

5. Suppose that X is normal and let C, D be two disjoint closed subsets of X. Prove that there is a continuous map $f : X \to \mathbb{R}$ so that $f(C) = 0$ and $f(D) = 1$.

6. Suppose that $p : X \to Z$ is a quotient map, and $f : X \to Y$ is continuous. Prove that there exists a continuous map $\overline{f} : Z \to Y$ with $\overline{f} \circ p = f$ if and only if f is constant on the fibers of p.

2 Part 2: Smooth Manifolds

1. Prove the local immersion theorem. If $F : M \to N$ is a smooth map of smooth manifolds and F_* is injective at P, then there are local coordinates (U, ϕ) at P and (V, ψ) at $F(P)$ so that $F(U) \subset V$ and

$$\psi \circ F \circ \phi^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0).$$

2. Let $SL_n(\mathbb{R})$ be the set of $n \times n$ matrices with real entries. Prove that $SL_n(\mathbb{R})$ is a smooth submanifold of $\mathbb{R}^{n^2}$ and identify the tangent space at the identity.

3. Prove that if $F : M \to N$ and $G : N \to X$ are smooth maps of smooth manifolds that $(G \circ F)_* = G_* \circ F_*$.

4. Let T^2 be the smooth surface in $\mathbb{R}^3$ that is the result of rotating the circle of radius $1/2$ in the yz-plane centered at $(0, 1, 0)$ about the z-axis. Let $p : T^2 \to \mathbb{R}^2$ be projection onto the xy-plane. Identify the regular values of p.

5. Let $P \in \mathbb{R}^n$ be a point. Prove that the vector space of derivations on the germs of smooth functions at P is isomorphic to $\mathbb{R}^n$.

6. Let $c : [0, 1]^k \to M$ be a smooth singular k-cube. Let ω be a smooth $k - 1$-form on M. Prove that

$$\int_c d\omega = \int_{\partial c} \omega.$$