# Practice Exam 4: Topology

June 2, 2005

*Directions: This is a three hour closed book exam. There are two parts of the exam. Work four questions from part 1 and four questions from part 2.*

## 1 Part 1: General Topology

1. Let $Y$ be Hausdorff and suppose $f, g : X \to Y$ are continuous. Prove that the set $\{x | f(x) = g(x)\}$ is a closed subset of $X$.

2. Prove that the continuous image of a compact set is compact. Prove that the continuous image of a connected set is connected.

3. Prove that any space $X$ is locally connected if and only if the connected components of each open set are open.

4. Prove that any regular second countable space is normal.

5. Prove that if $A, B$ are closed subsets of $X$ with $A \cup B = X$, and $f|_A$ and $g|_B$ are continuous, and $f$ and $g$ agree on $A \cap B$, then $f \cup g : X \to Y$ is continuous.

6. Prove that a metric space $X$ is compact if and only if it is sequentially compact.

## 2 Part 2: Smooth Manifolds

1. Let $SL_2(\mathbb{R})$ denote the Lie group of two by two real matrices of determinant 1. Write out formulas for the left invariant vector fields coming

from the tangent vectors at the identity $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then compute their Lie Bracket.

2. Let $V$ be a vector space and let $Alt : \mathcal{T}(V) \to \mathcal{A}(V)$ be the standard map that takes tensors to alternating tensors. Prove that the kernel of $Alt$ is an ideal. That is, if $Alt(T) = 0$ then for any tensor $S$,

$$Alt(T \otimes S) = 0.$$

3. Prove that any $C^\infty$- compatible atlas on a manifold $M$ is contained in a maximal compatible atlas.

4. Use stereographic projection from the north pole to define a coordinate patch on $S^2$. Compute the coefficients of the metric tensor induced from the inclusion in $\mathbb{R}^3$ in these coordinates.

5. Prove that the inclusion map $i : M \to N$ of a smooth submanifold is an immersion.

6. Prove that if $M$ is a smooth submanifold of $N$ then $f : M \to \mathbb{R}$ is smooth if and only if at each $P \in M$ there is $U$ open in $N$ containing $P$ and a smooth function $\overline{f} : U \to \mathbb{R}$ so that $\overline{f}|_{M \cap U} = f|_{M \cap U}$.