22M:017, Fall 2006
Lecture 12 (9/18/06).

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In the last lecture we discussed the definition of differentiability of a function at a point and the relationship of the derivative with the existence of a tangent line at a point of the graph of the function as well as the definition of instantaneous rate of change.

One observation that is valuable is that if a function f is differentiable at a point x_0 in an open interval (the only case we are considering) then:

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$

If $f'(x_0) > 0,$ then, when $h > 0$ is small:

$$\frac{f(x_0+h) - f(x_0)}{h} > 0$$
$$f(x_0+h) - f(x_0) > 0$$
$$f(x_0+h) > f(x_0).$$

If $h < 0$ and small in absolute value, then:

$$\frac{f(x_0+h) - f(x_0)}{h} > 0, h < 0$$
$$f(x_0+h) - f(x_0) < 0$$
$$f(x_0+h) < f(x_0).$$

The conclusion is that near x_0, f must be increasing relative to x_0.

If $f'(x_0) < 0,$ then the conclusion is (by the same type of discussion) that f is decreasing near x_0.

If $f'(x_0) = 0$ there is NO conclusion regarding these matters.

If I is an open interval and if $f : I \to \boldsymbol{R}$ is a function, we say that f is differentiable if $f'(x_0)$ exists for all $x_0 \in I.$ In this case we define the derivative function of $f, f'; I \to R,$ by the rule: if $x \in I$ then $f'(x)$ is the derivative of f at x.

Example. Consider the function $f : \boldsymbol{R} \to \boldsymbol{R}, f(x) = x^2.$ Find the derivative function of f.

Let $x \in \boldsymbol{R}.$ We need to find

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{(x+h)^2-x^2}{h}\\ &= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}\\ &= \lim_{h\to 0}\frac{2xh+h^2}{h}\\ &= \lim_{h\to 0}(2x+h)\\ &= 2x.\end{aligned}$$

Thus f is differentiable and $f' : \boldsymbol{R} \to \boldsymbol{R}, f'(x) = 2x$.

Differentiation Facts (which you must REMEMBER).

1. A constant function is differentiable and its derivative is zero at all points.

Example. If $f(x) = 2, x \in \boldsymbol{R}, f'(x) = 0$.

2. Power Rule.

If r is a real number and $f(x) = x^r$ then f is differentiable in its domain and $f'(x) = rx^{r-1}$. If r is zero, the function is constant (equal to one) and has zero derivative on R. If r is a positive integer, the domain is $\boldsymbol{R}$, if r is a negative integer, the domain is $(-\infty,0) \cup (0,\infty)$, and for all other values, except for special rational numbers such as $\frac{1}{3}, \frac{2}{3}, \frac{1}{5}$, etc.) the domain is considered to be the set of positive real numbers $(0,\infty)$.

Example. Let $f(x) = x^{100}$. Then f is differentiable on $\boldsymbol{R}$ and $f'(x) = 100x^{99}$.

Example. Let $f(x) = x^{\pi}$. Then f is differentiable on $\boldsymbol{R}$ and $f'(x) = \pi x^{\pi-1}$.

Example. Find

$$\lim_{h\to 0}\frac{(8+h)^{\frac{2}{3}}-4}{h}.$$

Here we have the definition of derivative used with the function $f(x) = x^{\frac{2}{3}}$ and $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$. So the limit is

$$\begin{aligned}f'(8) &= \frac{2}{3}8^{-\frac{1}{3}}\\ &= \frac{1}{3}.\end{aligned}$$

2. Constant Multiple Rule.

Let I be an open interval, $f : I \to \boldsymbol{R}$ a differentiable function and $\alpha \in \boldsymbol{R}$. Then αf is differentiable and

$$(\alpha f)'(x) = \alpha f'(x).$$

Example. Find the derivative of the function $f(x) = \sqrt{2}\,x^{10}$.

Here f is the product of $\sqrt{2}$ and the power function with exponent 10. Thus:

$$f'(x) = 10\sqrt{2}\,x^{9}.$$

3. Sum Rule.

Let I be an open interval and f, g two real valued differentiable functions defined on I. Then the sum function $f+g$ is differentiable on I and:

$$(f+g)'(x) = f'(x) + g'(x).$$

Example. The function $h(x) = \pi x^{\sqrt{2}} + \frac{1}{3}x^{\frac{5}{4}}$ is a differentiable function and

$$h'(x) = \sqrt{2}\,\pi x^{\sqrt{2}-1} + \frac{5}{12}x^{\frac{1}{4}}.$$

Example. Consider the function $f(x) = 3x^5 - 2x^4 + 5x^3 - 2x^2 + 6x - 5$. This is a differentiable function since it is the sum of differentiable functions and

$$f'(x) = 15x^4 - 8x^3 + 15x^2 - 4x + 6.$$

4. Product Rule.

Let f, g be differentiable functions on an open interval I. Then the product function fg is differentiable on I and:

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Observe that the derivative of the product IS NOT the product of the derivatives.

Example. Find the derivative of the function $f(x) = x^{\frac{2}{3}}(2x + x^{\pi})$.

Here the function we want to differentiate is the product of $x^{\frac{2}{3}}$. Thus f is differentiable and:

$$\begin{aligned} f'(x) &= (x^{\frac{2}{3}})'(2x + x^{\pi}) + x^{\frac{2}{3}}(2x + x^{\pi})' \\ &= \frac{2}{3}x^{-\frac{1}{3}}(2x + x^{\pi}) + x^{\frac{2}{3}}(2 + \pi x^{\pi-1}). \end{aligned}$$

Example. Find the derivative of the function $f(x) = (\frac{3}{5}x^5 + \frac{1}{4}x^4 - 2x^3 + 1)(-\frac{5}{4}x^4 + 2x^3 - 3x^2 + 2x - \pi)$. Find the derivative of this function.

In this case (and also in the previous example) we could multiply the terms and get a sum of differentiable functions and use the sum rule. When we learn to differentiate more complicated functions this will no longer be an option, and in the exam the alternatives may force you to use the product rule.

Using the product rule we get:

$$f'(x) = (3x^4 + x^3 - 6x^2)(-\frac{5}{4}x^4 + 2x^3 - 3x^2 + 2x - \pi) + (\frac{3}{5}x^5 + \frac{1}{4}x^4 - 2x^3 + 1)(-5x^3 + 6x^2 - 6x + 2).$$

5. Quotient Rule.

Let I be an open interval and f, g be differentiable functions defined on I. Then the quotient $\frac{f}{g}$, that has domain the set of points in I where g is not zero. On this domain the quotient function is differentiable and:

$$(\frac{f}{g})'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

$$(\frac{f(x)}{g(x)})' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

Example. Find the derivative of the function $q(x) = \frac{\sqrt{x}}{x^2+1}$.

In this function, the denominator is never zero. The numerator and denominator are differentiable functions on $(0, \infty)$. We use the quotient rule:

$$q'(x) = \frac{\frac{1}{2\sqrt{x}}(x^2 + 1) - \sqrt{x}\,(2x)}{(x^2 + 1)^2}.$$

Of course we could do the algebra to get a different expression for the derivative function.

6. Natural exponential function and natural logarithmic function.

$$(e^x)' = e^x$$

$$\ln'(x) = \frac{1}{x}.$$

Here we are giving you the fact that the derivative of the natural exponential function is the natural exponential function (the same function). This is one of the reasons why the term "natural " is used.

It also says that the derivative of the function $\ln : (0, \infty) \to \boldsymbol{R}$, is $\ln'(x) = \frac{1}{x}$.

Example. Find the derivative of the function $f(x) = x^2 e^x$.

Here we have to use the product rule as well as knowing that the derivative of e^x is e^x.

$$f'(x) = 2xe^x + x^2 e^x.$$

Example. Find the derivative of the function $f(x) = x^3 \ln(x)$.

$$f'(x) = 3x^2 \ln(x) + x^3 \frac{1}{x}$$

$$= 3x^2 \ln(x) + x^2.$$

Example. Fin the derivative of the function $g(x) = \frac{\ln(x)}{e^x+1}$.

Here we have to use the quotient rule:

$$g'(x) = \frac{\frac{1}{x}(e^x + 1) - \ln(x)e^x}{(e^x + 1)^2}.$$

What about other logarithmic functions (other bases). For this we recall that (and we MUST recall this): if $a > 0, a \neq 1$ then

$$\log_a(x) = \frac{\ln(x)}{\ln(a)} = \frac{1}{\ln(a)}\ln(x).$$

Hence $\log_a(x)$ is differentiable and:

$$\log_a'(x) = \frac{1}{\ln(a)x}.$$

Example. Find the derivative of the function $\log_2(x)$.

Here we find that $\log_2(x) = \frac{\ln(x)}{\ln(2)}$. Using the constant multiple rule, we get:

$$\log_2'(x) = \frac{1}{\ln(2)x}.$$

7. Chain Rule.

Let f, g be differentiable functions such that $f \circ g$ is defined. Then $f \circ g$ is differentiable and:

$$(f \circ g)'(x) = f'(g(x))g'(x)$$
$$(f(g(x)))' = f'(g(x))g'(x).$$

Example. Find the derivative of

$$f(x) = e^{x^2+x+1}.$$

From the Chain Rule:

$$f'(x) = e^{x^2+x+1}(2x + 1).$$

Consequences. Let $a > 0, a \neq 1$ and consider $f(x) = a^x = e^{\ln(a)x}$. Thus

$$(e^{\ln(a)x})' = \ln(a)e^{\ln(a)x}$$
$$(a^x)' = \ln(a)a^x.$$

Example. If $f(x) = 3^x, f'(x) = \ln(3)3^x = \ln(3)e^{\ln(3)x}$.

Example. Find the derivative of the function $f(x) = \pi^{x^3+x^2+x+1}$.

Here the power function is π^x and the function it is being composed with is $g(x) = x^3 + x^2 + x + 1$. Thus:

$$f'(x) = \ln(\pi)\pi^{x^3+x^2+x+1}(3x^2 + 2x + 1).$$