22M:017, Fall 2006
Lecture 17 (10/54/06)

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Announcement:

I apologize for the problems caused by my incorrect transcription of problems to Form C in the first exam

The exam will be regraded today giving all students credit for any answer in the corresponding problems in each form. This means that any answer will be accepted for the following problems

Form A: 8,10,19

Form B: 3,12,14

Form C: 8,17,19

Form D:8,10,13

This decision has the approval of the Chair of the Department of Mathematics.

The pop quizzes are used in the computation of the final grade only as attendance, no matter which is the score obtained. The information gathered about missed problems is used to guide discussion sections.

We have seen the implicit differentiation technique. We will do several examples today.

Example. Use implicit differentiation to find $\frac{dy}{dx}$ when x, y satisfy the equation:

$$x\ln(y) - yx^2 + \frac{3}{4e} = 0$$

Find $\frac{dy}{dx}$ at $(\frac{3}{2e}, e)$.

We have:

$$\begin{aligned}
x\ln(y) - yx^2 + \frac{3}{4e} &= 0\\
\ln(y) + \frac{x}{y}\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} &= 0\\
(\frac{x}{y} - x^2)\frac{dy}{dx} &= 2xy - \ln(y)\\
\frac{dy}{dx} &= \frac{2xy - \ln(y)}{\frac{x}{y} - x^2}.
\end{aligned}$$

We check that the point $(\frac{3}{2e}, e)$ is in the graph of the equation:

$$\begin{aligned} \frac{3}{2e}\ln(e) - e(\frac{3}{2e})^2 + \frac{3}{4e} &= \frac{3}{2e} - \frac{9}{4e} + \frac{3}{4e} \\ &= \frac{6-9+3}{4e} \\ &= 0. \end{aligned}$$

The point is in the graph of the equation. At $(\frac{3}{2e}, e)$, we have:

$$\begin{aligned} \frac{dy}{dx} &= \frac{2\frac{3}{2e}e - \ln(e)}{\frac{\frac{3}{2e}}{e} - (\frac{3}{2e})^2} \\ &= \frac{3-1}{\frac{3}{2} - \frac{9}{4e^2}} \\ &= \frac{8e^2}{6e^2 - 9}. \end{aligned}$$

Note that if the point was not in the graph of the equation then $\frac{dy}{dx}$ does not exist.

Example. Find $\frac{dy}{dx}$ for points in the graph if the equation where the derivative exist.

$$y\ln(x) + 1 = xe^y.$$

We have:

$$\begin{aligned} \frac{dy}{dx}\ln(x) + \frac{y}{x} &= e^y + xe^y\frac{dy}{dx} \\ (\ln(x) - xe^y)\frac{dy}{dx} &= e^y - \frac{y}{x} \\ \frac{dy}{dx} &= \frac{e^y - \frac{y}{x}}{\ln(x) - xe^y} \end{aligned}$$

Example. Find $\frac{dy}{dx}$, for points where it exists, in the graph of

$$x + \sqrt{xy} - y^2 = 0.$$

$$\begin{aligned} 1 + \frac{1}{2\sqrt{xy}}(y + x\frac{dy}{dx}) - 2y\frac{dy}{dx} &= 0 \\ (\frac{x}{2\sqrt{xy}} - 2y)\frac{dy}{dx} &= -(1 + \frac{y}{2\sqrt{xy}}) \\ \frac{dy}{dx} &= -\frac{1 + \frac{y}{2\sqrt{xy}}}{\frac{x}{2\sqrt{xy}} - 2y}. \end{aligned}$$

Example. At a certain factory, output Q is related to inputs x, y by the equation

$$Q = (\frac{1}{3})x^3 + \frac{1}{2}x^2y^2 + \frac{1}{9}(1+y)^3.$$

The current levels of input are $x = 4, y = 2,$ so $Q = (\frac{1}{3})4^3 + \frac{1}{2}4^2 2^2 + \frac{1}{9}(1+2)^3 = \frac{64}{3} + 32 + 3 = \frac{64+96+9}{3} = \frac{169}{3}.$

If the output is kept fixed, use the differential approximation formula to estimate the change of input y needed to offset a decrease of 0.8 units of x.

In this example the output is kept constant, at the value we found, so its derivative with respect to x is zero and the equation now relates only x and y.

$$(\frac{1}{3})x^3 + \frac{1}{2}x^2y^2 + \frac{1}{9}(1+y)^3 = \frac{169}{3}$$

$$x^2 + xy^2 + x^2y\frac{dy}{dx} + \frac{1}{3}(1+y)^2\frac{dy}{dx} = 0$$

$$16 + 16 + 32\frac{dy}{dx} + 3\frac{dy}{dx} = 0$$

$$35\frac{dy}{dx} = -32$$

$$\frac{dy}{dx} = -\frac{32}{35} \text{ when } x = 4, y = 2.$$

The differential approximation formula to be used is

$$y(3.2) - y(4) \cong \frac{dy}{dx} \times (-0.8).$$

Thus an estimate of the change needed is:

$$y(3.2) - y(4) \cong -\frac{32}{35} \times (-0.8)$$

$$= \frac{25.6}{35} \text{ units.}$$

Higher Order Derivatives.

Let I be an open interval and $f : I \to \boldsymbol{R}$ be a differentiable function. If the function $f' : I \to \boldsymbol{R}$ is also differentiable, then its derivative $(f')'$ is called the second derivative of f, and it is denoted by f'' or $\frac{d^2f}{dx^2}$. We say that f is twice differentiable.

Example. Consider the function $f : \boldsymbol{R} \to \boldsymbol{R}, f(x) = \frac{1}{3}x^3 - 5x^2 + 2x - \pi^\pi$. Find the second derivative of f.

$$f'(x) = x^2 - 10x + 2$$
$$f''(x) = 2x - 10.$$

Example. Consider the function $f : \boldsymbol{R} \to \boldsymbol{R}, f(x) = e^{-(x^2)}$. Find the second derivative of f.

$$\frac{df}{dx}(x) = -2xe^{-(x^2)}$$
$$\frac{d^2f}{dx^2} = -2e^{-(x^2)} - 2x(-2xe^{-(x^2)})$$
$$= e^{-(x^2)}(4x^2 - 2).$$

Example. Find the second derivative of the function $f : \boldsymbol{R} \to \boldsymbol{R}, f(x) = \frac{x^2}{x^2+x+1}$.

$$f'(x) = \frac{2x(x^2 + x + 1) - x^2(2x + 1)}{(x^2 + x + 1)^2}$$
$$= \frac{x^2 + 2x}{(x^2 + x + 1)^2}$$
$$f''(x) = \frac{(2x + 2)(x^2 + x + 1)^2 - 2(x^2 + 2x)(2x + 1)(x^2 + x + 1)}{(x^2 + x + 1)^4}$$

Now suppose that we have a twice differentiable function $f : I \to \boldsymbol{R}$. We say that f is three times differentiable if the function $f'' : I \to R$ is differentiable and the third derivative of $f, f''' = \frac{d^3f}{dx^3} = f^{(3)}$ is defined to be $f'''(x) = (f'')'(x)$.

Example. Let $f : \boldsymbol{R} \to \boldsymbol{R}, f(x) = xe^x$. Find the third derivative of f.

$$f'(x) = e^x + xe^x$$
$$f''(x) = e^x + e^x + xe^x = 2e^x + xe^x$$
$$f'''(x) = 2e^x + e^x + xe^x = 3e^x + xe^x$$
$$\frac{d^4f}{dx^4}(x) = 3e^x + e^x + xe^x.$$
$$= 4e^x + xe^x.$$

Example. Find the third derivative of the function $f : \boldsymbol{R} \to \boldsymbol{R}, f(x) = e^{-(x^2)}$.

$$
\begin{aligned}
f'(x) &= -2xe^{-(x^2)} \\
f''(x) &= -2e^{-(x^2)} + 4x^2e^{-(x^2)} \\
f'''(x) &= 4xe^{-(x^2)} + 8xe^{-(x^2)} - 8x^3e^{-(x^2)} \\
&= x(12 - 8x^2)e^{-(x^2)}.
\end{aligned}
$$

Of course now we can define the fourth derivative of a function $f, f^{(4)} = \frac{d^4f}{dx^4}$ as the derivative of $f''' = \frac{d^3f}{dx^3} = f^{(3)}$, and on and on. If a function is k times differentiable and its k –th derivative $f^{(k)} = \frac{d^kf}{dx^k}$ we say that f is $k+1$ times differentiable and we define the $k+1$ derivative of $f, f^{(k+1)} = \frac{d^{k+1}f}{dx^k}$ as the derivative of $f^{(k)}$, that is:

$$f^{(k+1)} = (f^{(k)})'.$$