22M:017, Fall 2006
Lecture 18

Juan A. Gatica

Notice: There are still some issues with the grading of Exam 1. These issues should be resolved by Monday.

Your Discussion Section TA has all the information regarding your exam. including which weere your answers and how you responded. Please see your TA if you want this information

Increasing and Decreasing Functions.

Let I be an interval and $f : I \to \boldsymbol{R}$ a function. We say that:
a) f is increasing on I if whenever $x_1 < x_2, x_1, x_2 \in I$ then

$$f(x_1) < f(x_2).$$

b) f is decreasing on I if whenever $x_1 < x_2, x_1, x_2 \in I$ then

$$f(x_1) > f(x_2).$$

In terms of the graph of f we see that when the function is increasing on I then the graph of f is rising (from left to right) over I. If f is decreasing on I then the graph of f is declining over I.

If we had to apply the definition of increasing (or decreasing) function every time we need to determine this property over an interval, we would have to be solving inequalities which may be difficult or impossible for us to resolve. Thus the following fact is extremely useful:

Let I be an OPEN interval and f a function that is differentiable on I. Then:
a) If $f'(x) > 0, x \in I$ then f is increasing on I.
b) If $f'(x) < 0, x \in I$ then f is decreasing on I.

Example. Find the intervals where $f(x) = x^2 + 5x + 6$ is increasing or decreasing

Thanks to the fact given above, we only have to determine on which open intervals $f'(x)$ is positive and on which intervals it is negative. Then we have to see what happens at the (real) endpoints of these intervals (if any).

$$f'(x) = 2x + 5.$$

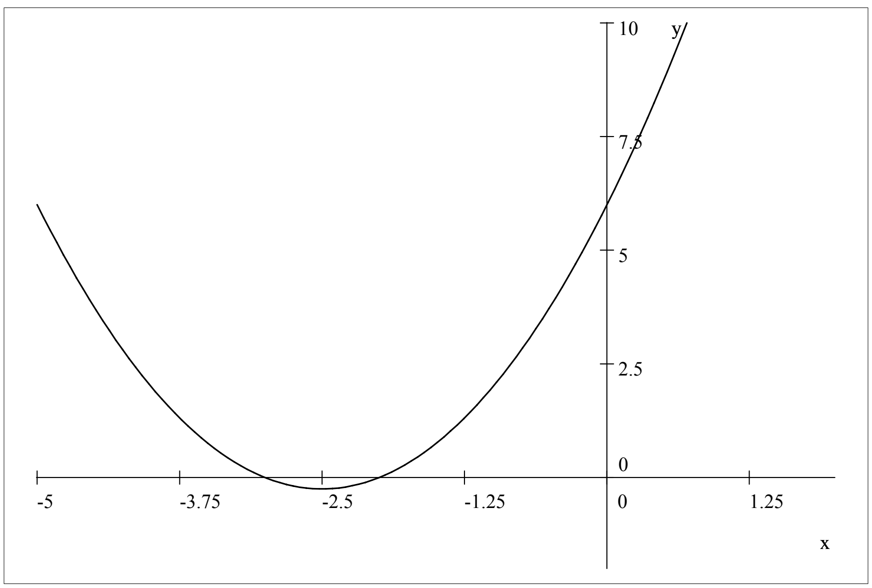
f will be increasing on the open intervals where

$$f'(x) > 0$$
$$2x + 5 > 0$$
$$x > -\frac{5}{2}.$$

Thus f is increasing on $(-\frac{5}{2}, \infty)$. What about on $[-\frac{5}{2}, \infty)$? Since f is continuous, f must be increasing on $[-\frac{5}{2}, \infty)$. Note that $f'(-\frac{5}{2}) = 0$.

The function will be decreasing on the interval $(-\infty, -\frac{5}{2})$ since its derivative is negative on this interval and, again the same reasoning tells us that f is decreasing on $(-\infty, -\frac{5}{2}]$.

The graph of f is:

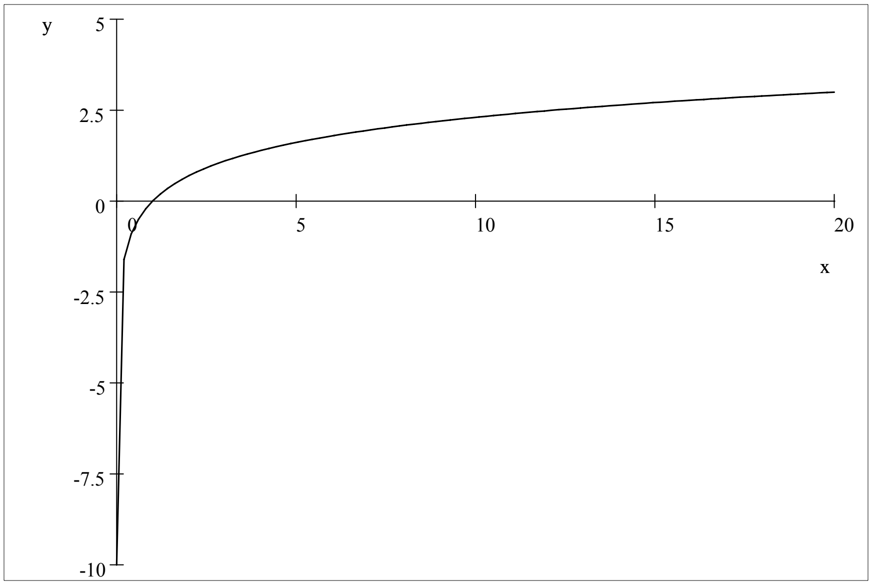


Plot of the function

Example. Find the intervals where the function $f(x) = \ln(x)$ is increasing or decreasing (if any).

$$f'(x) = \frac{1}{x}, x > 0.$$

Thus $f'(x) > 0$ on $(0, \infty)$ and is increasing there. This function is NOT increasing on $[0, \infty)$ since 0 is NOT in its domain.



Graph of $\ln$

Example. Find the intervals where the function $f(x) = xe^{-x^2}$ is increasing and those where it is decreasing.

$$f'(x) = e^{-x^2} - 2x^2e^{-x^2}$$
$$= (1-2x^2)e^{-x^2}.$$

Since $e^{-x^2} > 0,$ the derivative is positive where

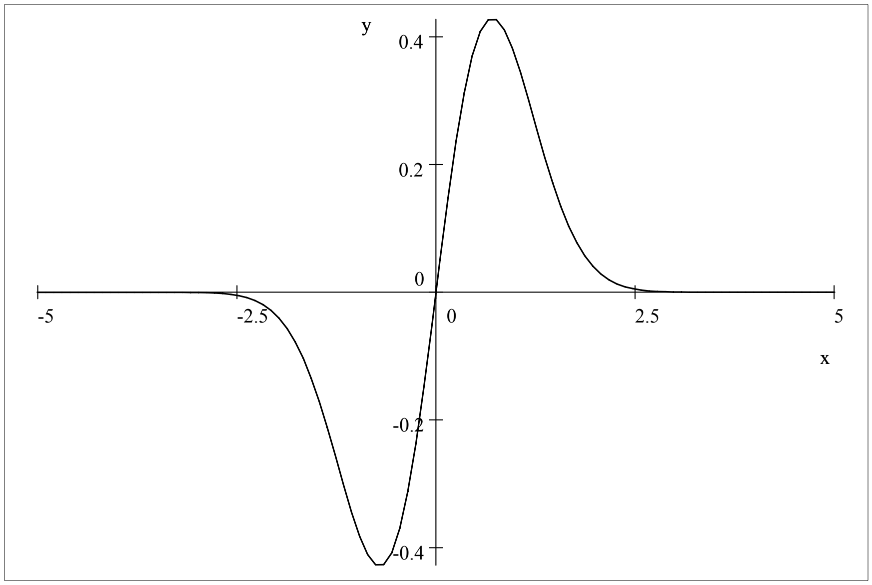
$$1 - 2x^2 > 0$$
$$x^2 < \frac{1}{2}$$
$$-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}.$$

The derivative is positive on $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Since the function is continuous we conclude that it is increasing on $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$.

The derivative is negative when

$$1 - 2x^2 < 0$$
$$1 < 2x^2$$
$$\frac{1}{2} < x^2$$
$$x < -\frac{1}{\sqrt{2}}, x > \frac{1}{\sqrt{2}}.$$

The function is decreasing on $(-\infty, -\frac{1}{\sqrt{2}}]$, and on $[\frac{1}{\sqrt{2}}, \infty)$.



Graph of $f(x) = xe^{-x^2}$

We observe that in some of these graphs there are points that are the "top of a hill " or the "bottom of a valley ". We explore the existence (or non-existence of) these points.

Definition Let I be an OPEN interval $f : I \to \boldsymbol{R}$ a function and $x_0 \in I$. We say that:

a) x_0 is a point of **relative** maximum in I if there is an **open** interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ such that it is contained in I and:

$$f(x) \leq f(x_0), x \in (x_0 - \varepsilon, x_0 + \varepsilon).$$

b) x_0 is a point of **relative** minimum in I if there is an **open** interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ such that it is contained in I and:

$$f(x) \geq f(x_0), x \in (x_0 - \varepsilon, x_0 + \varepsilon).$$

c) x_0 is a point of **relative extremum** if it is either a point of relative maximum or a point of relative minimum.

An endpoint of an interval cannot be a point of relative extremum in this interval.

Example. Determine if the function $f(x) = xe^{-x^2}$ has points of relative extremum and, if they exist, determine if they are points of relative maximum or points of relative minimum.

We have already studied the intervals where f is increasing or decreasing, concluding that it is increasing on $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$ and decreasing on $(-\infty, -\frac{1}{\sqrt{2}}]$, and on

$[\frac{1}{\sqrt{2}},\infty)$. Note that $f'(-\frac{1}{\sqrt{2}}) = f'(\frac{1}{\sqrt{2}}) = 0$.

Since the function is decreasing on $(-\infty,-\frac{1}{\sqrt{2}}]$ and increasing on $[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}]$, the conclusion is that $-\frac{1}{\sqrt{2}}$ is a point of relative minimum. Also, since it is increasing on $[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}]$ and decreasing on $[\frac{1}{\sqrt{2}},\infty)$, $\frac{1}{\sqrt{2}}$ is a point of relative maximum.

We know that if a function is differentiable at a point x_0 then, if $f'(x_0) > 0$ it is increasing at x_0 and if $f'(x_0) < 0$, it is decreasing at this point. Thus if x_0 is a point of relative extremum then either $f'(x_0) = 0$ or f is NOT differentiable at x_0.

Definition. Let I be an open interval, $f : I \to \boldsymbol{R}$ a **continuous** function and $x_0 \in I$. We say that x_0 is **a critical point** of f if either f is NOT differentiable at x_0 OR f is differentiable at x_0 and $f'(x_0) = 0$.

Important observation. If x_0 is a point of extremum for f then it must be a critical point. Thus we search for points of extremum only among the critical points. However, if x_0 is a critical point of f then it NEED NOT be a point of extremum. For example, if $f(x) = x^3$ then $f'(x) = 3x^2$, so that 0 is a critical point but f is increasing on $(-\infty,0\}$ and on $[0,\infty)$ (since its derivative is positive at all non-zero real numbers), concluding that f is increasing on $\boldsymbol{R}$.

Another relevant example of this phenomenon is given by the function $f(x) = x^{\frac{1}{3}}$. Then $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}, x \neq 0$. f is NOT differentiable at 0 (it is a continuous function), so 0 is a critical point since f is not differentiable at this point, but since $f'(x) > 0, x \neq 0$ again f is increasing on $\boldsymbol{R}$.

Now suppose that we have a function $f : I \to \boldsymbol{R}$ and we find a critical point x_0 for f. How can we determine if it is a point of local maximum or a point of local minimum or if it is not a point of relative extremum? The first answer to this question is:

First derivative test: If x_0 is a critical point of f then:

a) If there is an interval of the form $(x_0 - \varepsilon, x_0)$ where $\varepsilon > 0$ and $f'(x) > 0$ (so f is increasing on this interval) and another interval of the form $(x_0, x_0 + \delta)$ where $\delta > 0$ and $f'(x) < 0$ (so f is decreasing on this interval), then x_0 is a point of relative maximum.

b) If there is an interval of the form $(x_0 - \varepsilon, x_0)$ where $\varepsilon > 0$ and $f'(x) < 0$ (so f is decreasing on this interval) and another interval of the form $(x_0, x_0 + \delta)$ where $\delta > 0$ and $f'(x) > 0$ (so f is increasing on this interval), then x_0 is a point of relative minimum.

Example. Find all critical points of the function $f(x) = e^{x^2-x+6}$ and determine if they are points of local extremum. If the are points of local extremum, determine their nature (maximum or minimum).

$$f'(x) = (2x-1)e^{x^2-x+6}.$$

Thus f is differentiable at all points so the only critical points are those where the derivative is zero. Thus the only critical point is $x = \frac{1}{2}, f'(x) < 0$ for $x < \frac{1}{2}, f'(x) > 0, x > \frac{1}{2}$. $x = \frac{1}{2}$ is a point of relative minimum.

Example. Find all critical points for $f(x) = (x^2+5x+6)^{\frac{1}{3}}$, and determine if they are points of local extrema, and, if so, the nature of the point.

$$\begin{aligned} f'(x) &= \frac{1}{3}(x^2+5x+6)^{-\frac{2}{3}}(2x+5) \\ &= \frac{1}{3}((x+2)(x+3))^{-\frac{2}{3}}(2x+5). \end{aligned}$$

The critical points where the derivative does not exist are $x = -3, x = -2.$ Also $x = -\frac{5}{2}$ is a point where the derivative is zero, so it is a critical point.

If $x < -3, f'(x) < 0,$ if $-3 < x < -\frac{5}{2}, f'(x) < 0.$ $x = -3$ is NOT a point of local extremum.

If $-\frac{5}{2} < x < -2, f'(x) > 0.$ $x = -\frac{5}{2}$ is a point of local minimum.

If $x > -2, f'(x) > 0,$ so $x = -2$ is NOT a point of local extremum.

Thus $x = -\frac{5}{2}$ is the only point of local extremum and it is a point of local minimum.

$f(x) = (x^2+5x+6)^{\frac{1}{3}}$