22M:017, Fall 2006
Lecture 19

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Announcement: Due to the fact that some of the Teaching Assistants are having midterms themselves, the corrected grades should appear by Wednesday October 11.

We have discussed how the first derivative helps in finding out when differentiable functions are increasing or decreasing, and how they are used in determining when a critical point is a point of local extremum.

When dealing with profit, loss costs and so on, one common approach is to look at the average functions related to these issues.

Suppose that we have a situation where we know the cost function $C(x)$ for producing x items and that this is a differentiable function. Then the average cost function $AC(x)$ is simply

$$AC(x) = \frac{C(x)}{x}.$$

The marginal cost function is $C'(x)$.

It makes sense to try to find if there is a minimum for the average cost function. If there is a minimum then it must be a critical point, so we differentiate this function:

$$AC'(x) = \frac{xC'(x) - C(x)}{x^2}.$$

The critical points are, therefore, the solutions to the equation

$$xC'(x) - C(x) = 0.$$

Thus any critical point x is such that

$$C'(x) = \frac{C(x)}{x}.$$

Therefore, we can say that the average cost function is minimized (if it can be minimized) at a point where the average cost equals the marginal cost. To determine if it is indeed a point of minimum, further analysis must be made.

Example. A sugar refinery can produce x tons of sugar per week at a weekly cost of

$C(x) = 0.1x^2 + 5x + 2250$ dollars. Find the level of production for which the average cost is at a minimum.

Solution.

$$\begin{aligned} AC(x) &= \frac{0.1x^2 + 5x + 2250}{x} \\ &= 0.1x + 5 + \frac{2250}{x}. \end{aligned}$$

Differentiating:

$$AC'(x) = 0.1 - \frac{2250}{x^2}.$$

The critical points, where the average cost equals the marginal cost are:

$$\begin{aligned} 0.1 - \frac{2250}{x^2} &= 0 \\ 0.1x^2 &= 2250 \\ x^2 &= 22500, x > 0 \\ x &= \sqrt{22500} \\ x &= 150 \end{aligned}$$

Thus $x = 150$ is the only candidate for a point. Of course we know that just having a critical point does not mean that we have a point of minimum.

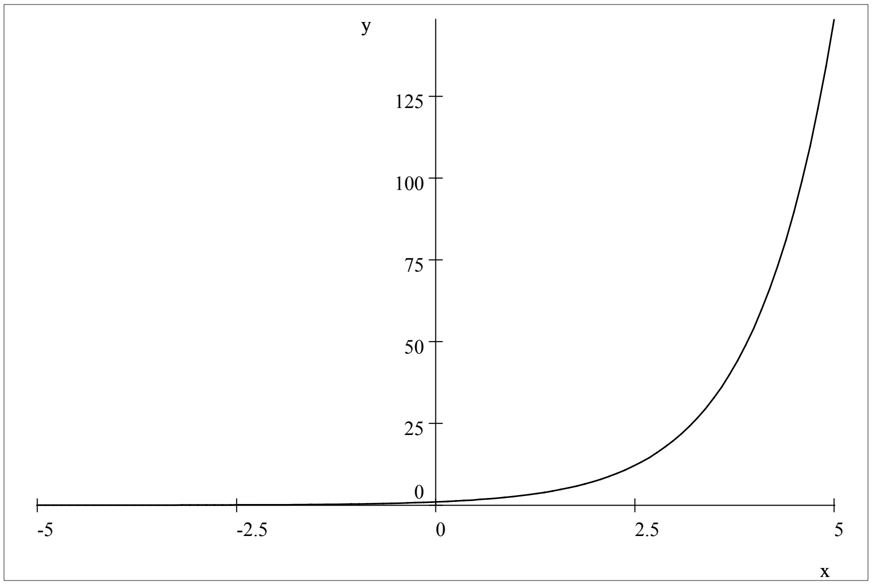
The only way we have right now to determine if it is indeed a point of minimum is by analyzing the sign of the first derivative of $AC(x), AC'(x)$:

$$\begin{aligned} AC'(x) &= 0.1 - \frac{2250}{x^2} \\ &= \frac{0.1x^2 - 2250}{x^2} \\ &= \frac{0.1(x^2 - 22500)}{x^2} \\ &= \frac{0.1(x - 150)(x + 150)}{x^2}. \end{aligned}$$

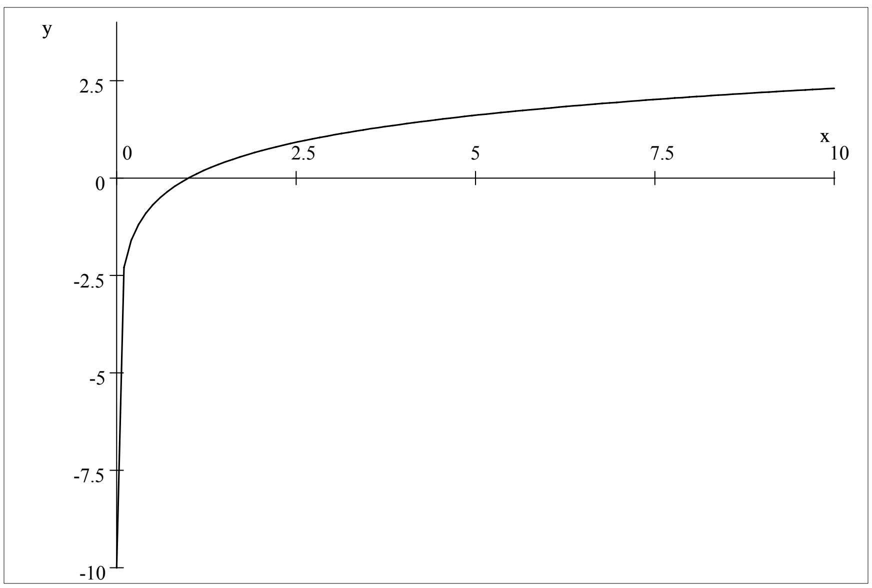
We see that if $0 < x < 150, AC'(x) < 0,$ and if $150 < x, AC'(x) > 0.$ This shows that $x = 150$ is the point of minimum for the average cost function. Also at a production level of 150 tons per week the average cost is equal to the marginal cost.

Now we will explore how the second derivative can be used to find other important properties of graphs of functions and also how they can be used to determine, when possible, the nature of a critical point.

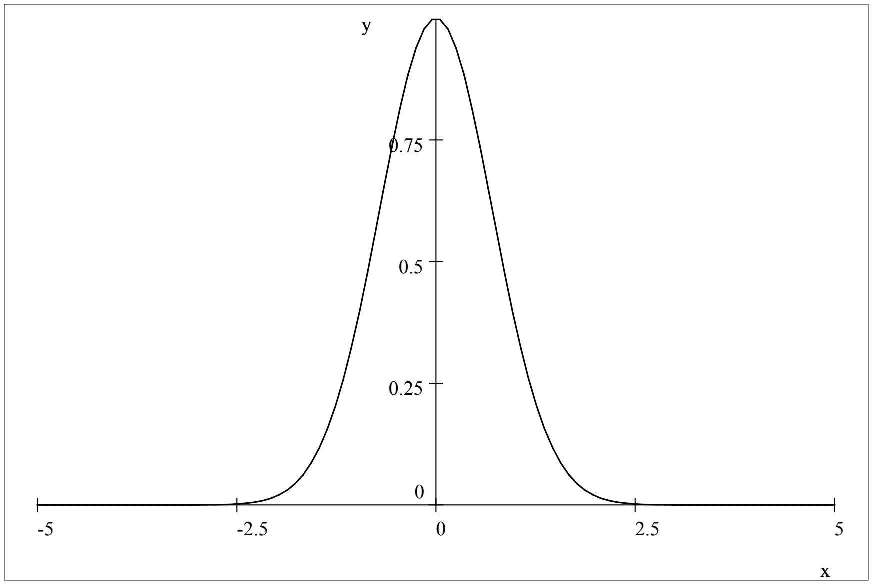
The graph of $f(x) = e^x$ is, as we know, the following:



We observe that is always "cupping " or "curving " up.
The graph of $g(x) = \ln(x)$, we recall:



This graph is always "cupping " down.
We also look at the graph of $h(x) = e^{-x^2}$.



Here we see that there are intervals over which the graph "cups " up and intervals over which it "cups "down.

In all cases, if we look at the slope of the tangent lines we get the information about the way the graph is curving. Instead of cupping " we will call the property "concavity".

Definition. Let I be an open interval and $f : I \to \boldsymbol{R}$ a differentiable function. We say that the graph of f is:

a) Concave up on I if f', the derivative of f is an increasing function on I.

b) Concave down over I if f' is a decreasing function on I.

We have seen how the derivative of a function allows us to find where he function is increasing and where it is decreasing. It is then natural to use this tool to decide where the function f' is increasing or decreasing by using its derivative $(f')' = f''$, the second derivative of f.

Test for Concavity.

Let I be an open interval and $f : I \to \boldsymbol{R}$ a twice differentiable function. Then:

a) If $f''(x) > 0, x \in I$ the graph of f is concave up over I.

b) If $f''(x) < 0, x \in I$ the graph of f is concave down over I.

Example. The function $f : \boldsymbol{R} \to \boldsymbol{R}, f(x) = e^x$ is such that $f'(x) = e^x, f''(x) = e^x > 0$. Thus its graph is concave up over $\boldsymbol{R}$.

Example. The function $\ln : (0,\infty) \to \boldsymbol{R}$ is such that $\ln'(x) = \frac{1}{x}, \ln''(x) = -\frac{1}{x^2} < 0$.

The graph of $\ln$ is concave down over $(0,\infty)$.

Example. Consider the function $f:\boldsymbol{R}\to\boldsymbol{R}, f(x)=e^{-x^2}$. Then:

$$\begin{aligned} f'(x) &= -2xe^{-x^2} \\ f''(x) &= -2e^{-x^2}+4x^2e^{-x^2} \\ &= 4(x^2-\frac{1}{2})e^{-x^2} \\ &= 4(x-\frac{1}{\sqrt{2}})(x+\frac{1}{\sqrt{2}})e^{-x^2}. \end{aligned}$$

We see that:

$$f''(x)>0 \text{ on } (-\infty,-\frac{1}{\sqrt{2}}) \text{ and on } (\frac{1}{\sqrt{2}},\infty)$$
$$f''(x)<0 \text{ on } (-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}).$$

The conclusion is that the graph of f is concave up over $(-\infty,-\frac{1}{\sqrt{2}})$ and over $(\frac{1}{\sqrt{2}},\infty)$. It is concave down over $(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$. The graph of f changes concavity at the points $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$.

Definition. Let I be an open interval and $f:I\to\boldsymbol{R}$ a differentiable function. We say that a point $x_0\in I$ is an inflection point (for the graph of f) if there exist numbers α,β such that $\alpha<x_0<\beta$ and the graph of f has one type of concavity over (α,x_0) and the OTHER type of concavity over (x_0,β).

In the example of $f(x)=e^{-x^2}$ the points $-\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ are points of inflection.

Important Fact: Inflection points for the graph of a differentiable function f are among the critical points of the derivative of f, f' (where $f''(x_0)=0$ or does not exist). However, not every critical point of f' is necessarily a point of inflection.

Example. Let $f:\boldsymbol{R}\to\mathbf{R}, f(x)=\ln(x^4+2)$. Discuss the concavity of the graph of the function and determine if there are any points of inflection.

$$f'(x) = \frac{4x^3}{x^4 + 2}$$

$$f''(x) = \frac{12x^2(x^4 + 2) - 4x^3 \times 4x^3}{(x^4 + 2)^2}$$

$$= \frac{24x^2 - 4x^6}{(x^4 + 2)^2}$$

$$= 4x^2 \frac{6 - x^4}{(x^4 + 2)^2}$$

$$= 4x^2 \frac{((6)^{1/2} - x^2)((6)^{1/2} + x^2)}{(x^4 + 2)^2}$$

$$= 4x^2 \frac{((6)^{1/4} - x)((6)^{1/4} + x)((6)^{1/2} + x^2)}{(x^4 + 2)^2}$$

The critical points of f' are:

$$x = -(6)^{1/4}, x = 0, x = (6)^{1/4}.$$

An examination of the signs of the second derivative yields:

$$x < -\sqrt[4]{6}$$

$$(6)^{1/4} - x > 0, (6)^{1/4} + x < 0$$

$$f''(x) < 0$$

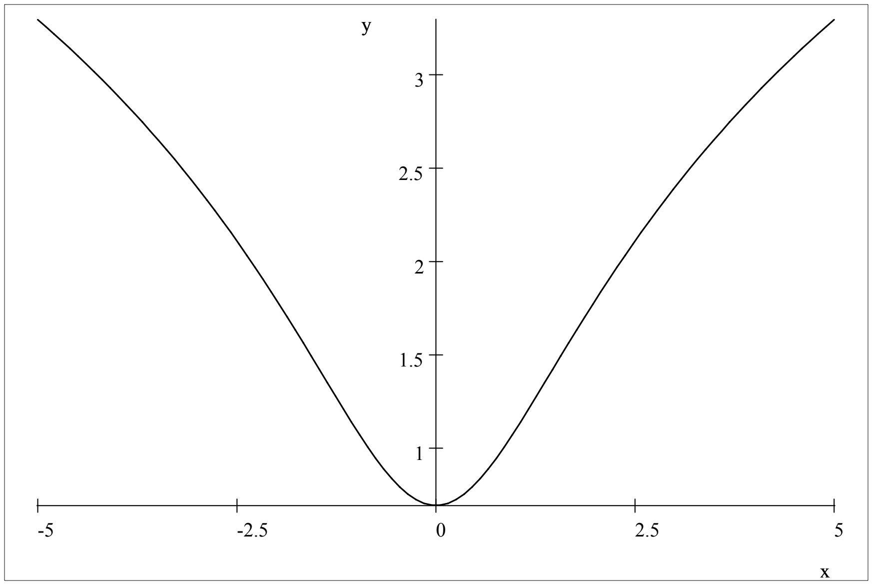
$$-\sqrt[4]{6} < x < 0, f''(x) > 0$$

$$0 < x < \sqrt[4]{6}, f''(x) > 0$$

$$x > \sqrt[4]{6}, f''(x) < 0$$

$f''(x) < 0$ on $(-\infty, -(6)^{1/4})$ and on $(\sqrt[4]{6}, \infty)$ The graph is concave down on these intervals.
$f''(x) > 0$ on $(-(6)^{1/4}, 0)$ and on $(0, (6)^{1/4})$. The graph is concave up over these intervals.

Note that 0 is a critical point of f' since $f''(0) = 0$ but is NOT a point of inflection since there is no change in concavity. The points $-(6)^{1/4}, (6)^{1/4}$ are points of inflection. Below is a sketch of the graph of f.



Now we concentrate on the nature of critical points for functions that are twice differentiable. Since we are assuming that the functions under consideration are twice differentiable, their graphs will be "smooth ", so the critical points of f will all have the derivative being zero at the point in question.

Now recall that if we have a point of local maximum, then the graph of f over an open interval containing the point should look like the "top of a hill ", so it should be concave down on this interval. On the other hand, if it is a point of local minimum, then we are dealing with the "bottom of a valley "and the concavity should be upwards. Therefore, we have;

Second Derivative Test for Local Extrema.

Let I be an open interval, $f : I \to \boldsymbol{R}$ a twice differentiable function and $x_0 \in I$ a critical point of f. Then:

a) If $f''(x_0) < 0, x_0$ is a point of local maximum.

b) If $f''(x_0) > 0, x_0$ is a point of local minimum.

c) If $f''(x_0) = 0$ the test is INCONCLUSIVE (no information is obtained through the second derivative).

Example. Consider the function $f : R \to R, f(x) = 3x^5 - 5x^3 + 2.$ Find all critical points and determine their nature.

$$f'(x) = 15x^4 - 15x^2$$
$$f''(x) = 60x^3 - 30x.$$

Critical points:

$$\begin{aligned} f'(x) &= 15x^4 - 15x^2 \\ &= 15x^2(x^2-1) \\ &= 15x^2(x-1)(x+1). \end{aligned}$$

Thus the critical points are $-1, 0, 1$.

$f''(-1) = -60 + 30 = -30 < 0, -1$ is a point of local maximum.

$f''(0) = 0,$ the second derivative test is inconclusive.

$f''(1) = 30 > 0, 1$ is a point of local minimum.

Now to find the nature of the critical point 0 we use the first derivative test. From the factorization of $f'(x)$ we see that:

$$f''(x) < 0, -1 < x < 0$$

$$f''(x) < 0, 0 < x < 1.$$

Thus 0 is NOT a point of local extremum.

Example. Find all critical points of $f(x) = xe^{-x^2}$ and determine their nature.

$$f'(x) = e^{-x^2} - 2x^2e^{-x^2}.$$

Critical points:

$$-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}.$$

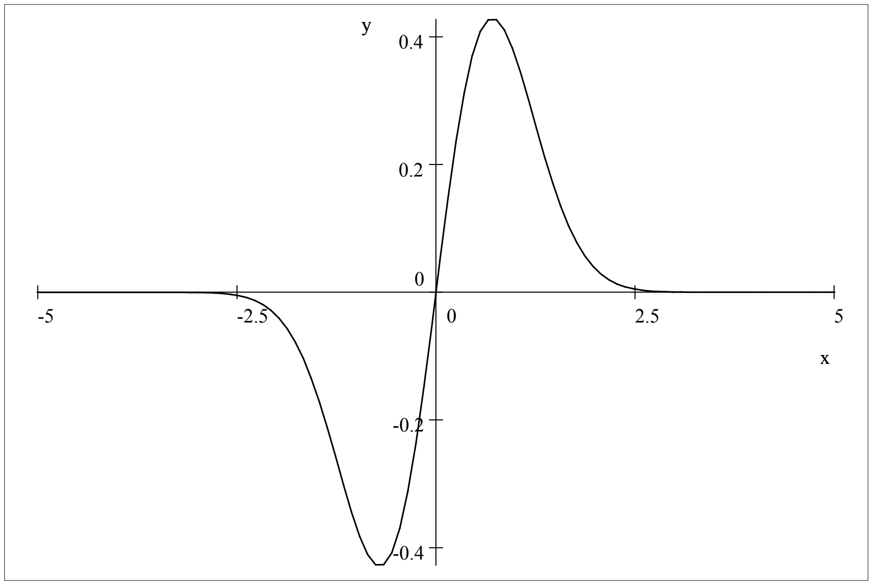
$$\begin{aligned} f''(x) &= -2xe^{-x^2} - 4xe^{-x^2} + 4x^3e^{-x^2} \\ &= -6xe^{-x^2} + 4x^3e^{-x^2} \end{aligned}$$

$$f''(-\frac{1}{\sqrt{2}}) = \frac{6}{\sqrt{2}}e^{-\frac{1}{2}} - \frac{2}{\sqrt{2}}e^{-\frac{1}{2}} > 0.$$

$-\frac{1}{\sqrt{2}}$ is a point of local minimum.

$$\begin{aligned} f''(\frac{1}{\sqrt{2}}) &= -\frac{6}{\sqrt{2}}e^{-\frac{1}{2}} + \frac{2}{\sqrt{2}}e^{-\frac{1}{2}} \\ &= \frac{e^{-\frac{1}{2}}}{\sqrt{2}}(2-6) < 0. \end{aligned}$$

$\frac{1}{\sqrt{2}}$ is a point of local maximum.


y
0.4
0.2
0
-0.2
-0.4
-5
-2.5
0
2.5
5
x