22M:017, Fall 2006
Lecture 2 (8/23/06)
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Notice: Due to time conflicts, the Add/Drop hours will be from 3:30 to 4:30 Monday, Wednesday and Friday, 10:30 to 11:30 Tuesdays and Thursdays, August 23 through September 1. Place: Math Lab, 314 MLH.

The websites for this course are: for syllabus extra homework problems and current lecture notes, www.math.uiowa.edu/~gatica, for the syllabus and the lecture notes from last year, www.math.uiowa.edu/mathlab.

In many situations you will be given a function by means of a formula, without specifying its domain. In this setting we will agree that its domain is the largest possible set of real numbers where the formula defines a function.

Example 6. Find the domain of the function

$$y = \frac{1}{x^2 - 4}.$$

In this case we must remember that we cannot divide by zero, so the domain must be the set of all real numbers x such that $x^2 - 4 \neq 0$. We have:

$$x^2 - 4 = (x+2)(x-2).$$

Thus the domain D is:

$$D = \{x \in \boldsymbol{R} : x \text{ is different from } -2 \text{ and } 2\}$$
$$D = (-\infty, -2) \cup (-2, 2) \cup (2, \infty).$$

If $x = -3, y = \frac{1}{5}, x = -1, y = -\frac{1}{3}, x = 1, y = -\frac{1}{3}, x = 4, y = \frac{1}{12}$.

Example 7. Find the domain of the function

$$v = \sqrt{t^2 - t - 6}.$$

In this case we first remember that the names of the variables is not important. We need

$$t^2 - t - 6 \geq 0$$
$$(t+2)(t-3) \geq 0.$$

Now we recall that he product of two real numbers is nonnegative if and only if both are nonnegative or both are less than or equal to zero, so we must have:

$$t \geq 3 \text{ or } t \leq -2.$$

The domain D is the set

$$D = \{t \in \boldsymbol{R} : t \leq -2 \text{ or } t \geq 3$$
$$D = (-\infty, -2] \cup [3, \infty).$$

$v(-2) = 0, v(-3) = \sqrt{6}, v(3) = 0, v(5) = \sqrt{14}.$

Example 8. Find the domain of the function

$$y = \frac{x}{\sqrt{10 - x^2 + 7x}}.$$

in this case we observe that, since division by zero is not defined, we must have

$$10 - x^2 + 7x > 0$$
$$x^2 - 7x - 10 < 0.$$

For this we need to use the quadraticc formula in order to factorize the quadratic polynomial.

$$x^2 - 7x - 10 = 0$$
$$x = \frac{7 \pm \sqrt{49 + 40}}{2}$$
$$x = \frac{7 \pm \sqrt{89}}{2}.$$

Then we have:

$$x^2 - 7x - 10 = (x - \frac{7 - \sqrt{89}}{2})(x - \frac{7 + \sqrt{89}}{2}).$$

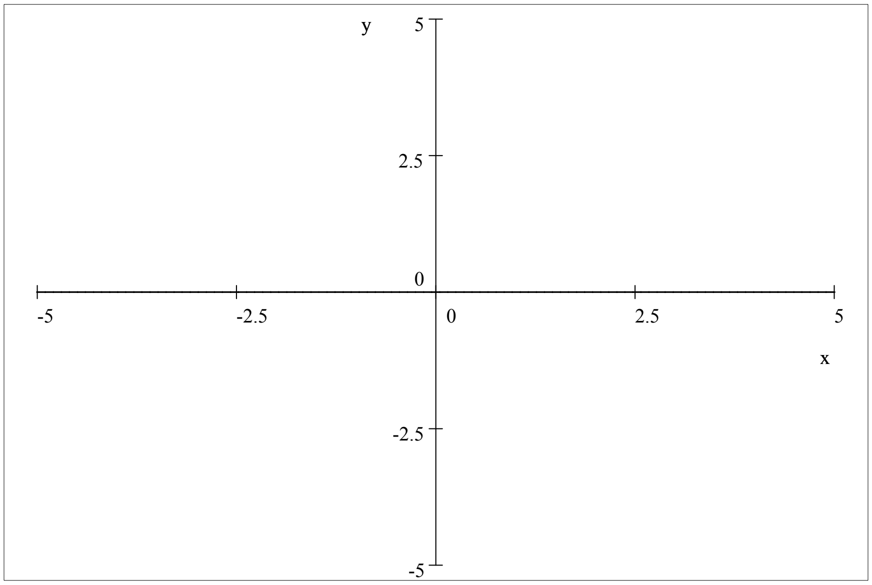
The domain D is:

$$D = (\frac{7 - \sqrt{89}}{2}, \frac{7 + \sqrt{89}}{2}).$$

Now we will start trying to graph functions. Right now we will only be able to plot a few points and "infer "the graph, with almost all the details of the graph being "guessed at "; after we discuss differential calculus we will be able to **know** the relevant properties of the graphs of many functions and the process of graphing will become much more of a science, as opposed to relying in our guesses.

The first step is to recall how Cartesian coordinates are used to describe the plane, that is to represent all points in the plane with ordered pairs of real numbers.

To do this, we draw two perpendicular straight lines . Zero is chosen as the point of intersection for each of the lines, a unit of length is chosen for each line, the real numbers are placed on each line, the horizontal line with the direction of increase being wet to east, the vertical line having direction of increase south to north. For any point P in the plane we draw a vertical line trough it and the real number on the horizontal line chosen at the start is the first component (abscissa) of the ordered pair of real numbers representing the point, we draw a horizontal line through P and the real we get in the vertical axis is the second coordinate of the ordered pair of real numbers representing the point (the ordinate). The picture we get is:



If P is a point in the plane, we draw through it a vertical line (parallel to the vertical axis) and the point in the x-axis where the intersection occurs is a real number x_0, then we draw through the point P and find the point of intersection with the vertical axis at a number y_0. The point P is identified with the ordered pair (x_0,y_0). Thus $(1,1),(-2,3),(-\sqrt{2},-\pi)$ are points in the plane. The plane is now identified with the set of ordered pairs of real numbers, dented by $\boldsymbol{R}^2$. $(0,0)$ 9s called the origin.

$$\boldsymbol{R}^2 = \{(x,y) : x \in \boldsymbol{R}, y \in \boldsymbol{R}\}$$

If X is a set of real numbers and $f : X \to \boldsymbol{R}$ is a function, then the set

$$G(f) = \{(x,y) \in \boldsymbol{R}^2 : x \in X, y = f(x)\}$$

or its picture in the plane, is called the graph of f.

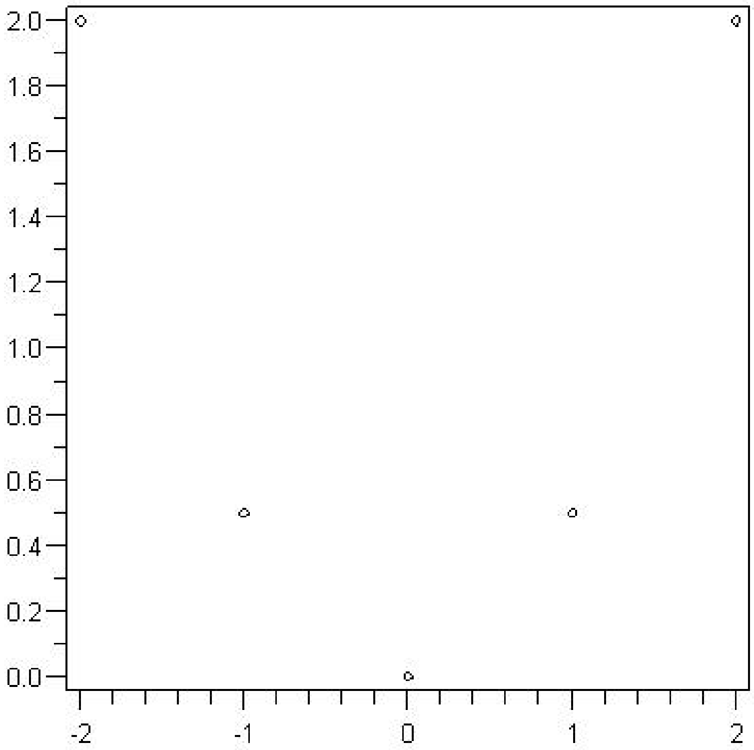
To draw the graph of f the only tool we have so far is to pick several points in the domain of the function, evaluate the function there, plot these points in the plane and guess the shape of the graph from this. This is a very unsatisfying procedure but it is all we have for now; when we have some calculus, we will be able to find properties of the graph that will take out the guess work for the functions considered in this course.

Exampl1 1. Graph $f : \boldsymbol{R} \to \boldsymbol{R}, f(x) = \frac{1}{2}x^2$.

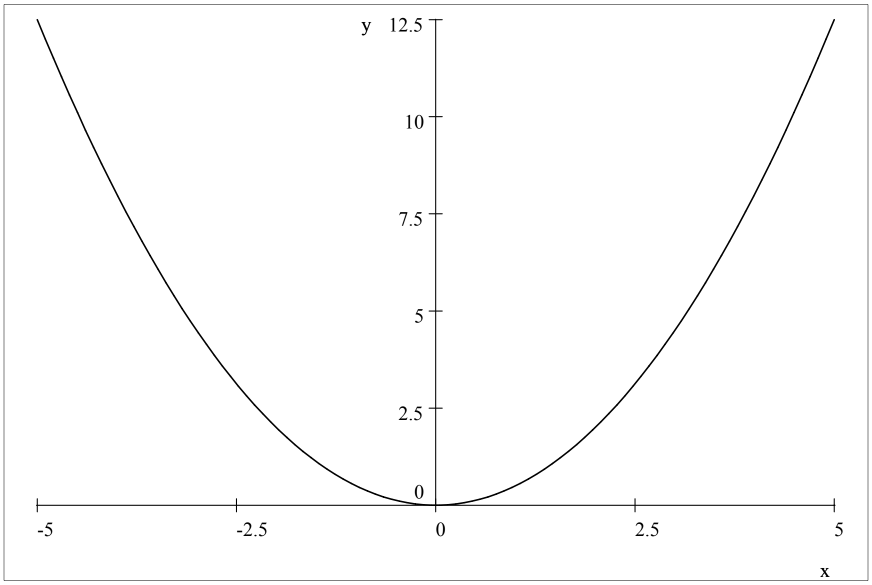
Following the procedure outlined, we make the following table of values of f to determine some points in the graph (if you already know that this is a parabola that opens up, GREAT).

$$\begin{bmatrix} x & f(x) \\ 0 & 0 \\ -1 & \frac{1}{2} \\ 1 & \frac{1}{2} \\ 2 & 2 \\ -2 & 2 \end{bmatrix}$$

Now we plot these points in the plane:



Now we plot the graph:



Graph of $f(x) = \frac{1}{2}x^2$

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If we have $a \neq 0, b, c \in \boldsymbol{R}$, we consider the function $f : \boldsymbol{R} \to \boldsymbol{R}$ given by:

$$f(x) = ax^2 + bx + c.$$

We would like to find a reasonable graph for it, so we will compare it with the graph of $g(x) = x^2$. We recall that:

$$(x+k)^2 = x^2 + 2kx + k^2.$$

We use this by "completing the square " below:

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x) + c \text{ (we factored } a) \\ &= a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}) + c \text{ (we added and subtracted } \frac{b^2}{4a^2}) \\ &= a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c \text{ (we took out one term in the parenthesis)} \\ &= a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a} \text{ (we re-wrote the term not in the parenthesis).} \end{aligned}$$

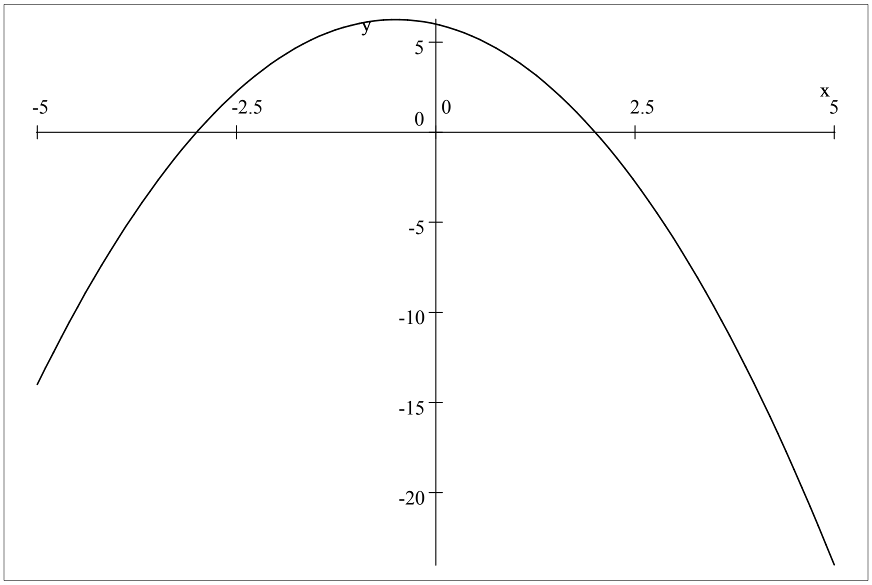
Not it is clear how the graph relates to the graph of g: We multiply the square term by a (it will open up if $a > 0$, down if $a < 0$), and it may be translated both vertically and/or horizontally (depending on a, b, c).

The "vertex "(highest point in the parabola if it opens down, lowest if it opens up) is the point $(-\frac{b}{2a}, -\frac{b^2-4ac}{4a})$.

Example 2. Find the vertex, determine if the parabola opens up or down, and plot:

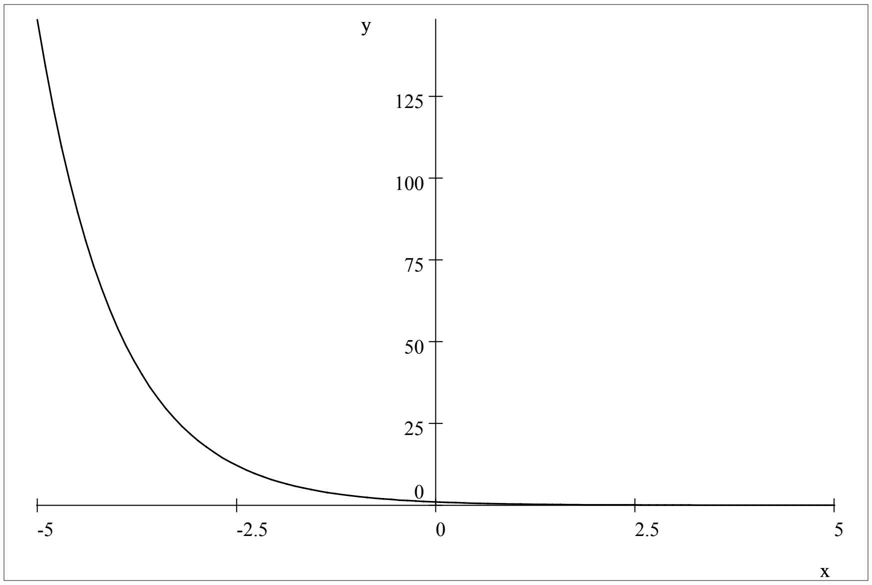
$$y = -x^2 - x + 6.$$

Here $a = -1, b = -1, c = 6, \frac{b^2-4ac}{4a} = \frac{1+24}{-2} = -\frac{25}{2}$, vertex $(-\frac{-1}{-2}, -(-\frac{25}{2})) = (-\frac{1}{2}, \frac{25}{2})$. It opens down.

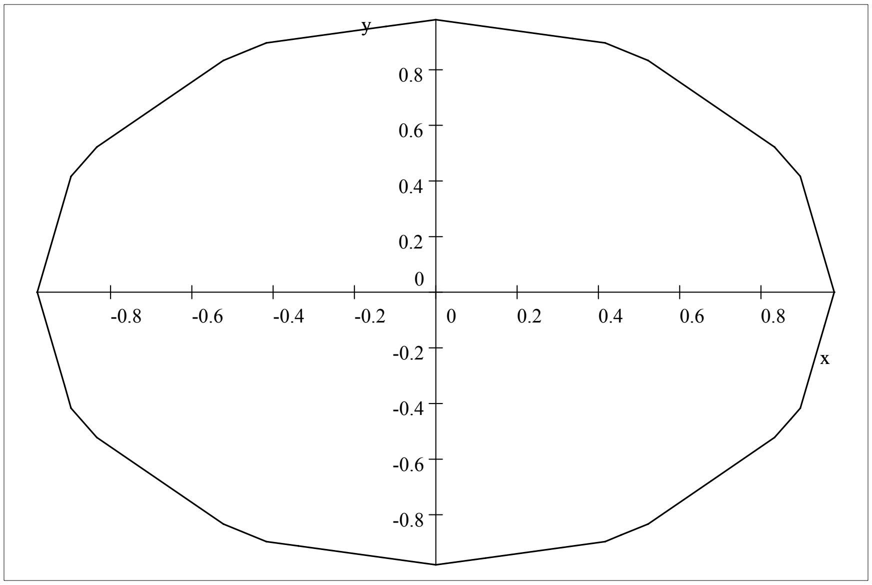


Grap of $f(x) = -x^2 - x + 6$

If we have a curve in the plane we can easily tell if it is the graph of a function, base on the definition of function that states the each point in the domain is associated wit one and only one element in the range, giving us the vertical line test: a curve in the plane is the graph of a function if and only if each vertical line crosses the curve in AT MOST one point.



Curve that is the graph of a function



Not the graph of a function

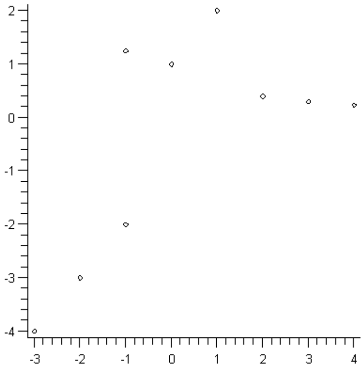
Example 3. Plot the graph of $f : \boldsymbol{R} \to \boldsymbol{R}$,

$$f(x) = x - 1, x \leq -1,$$

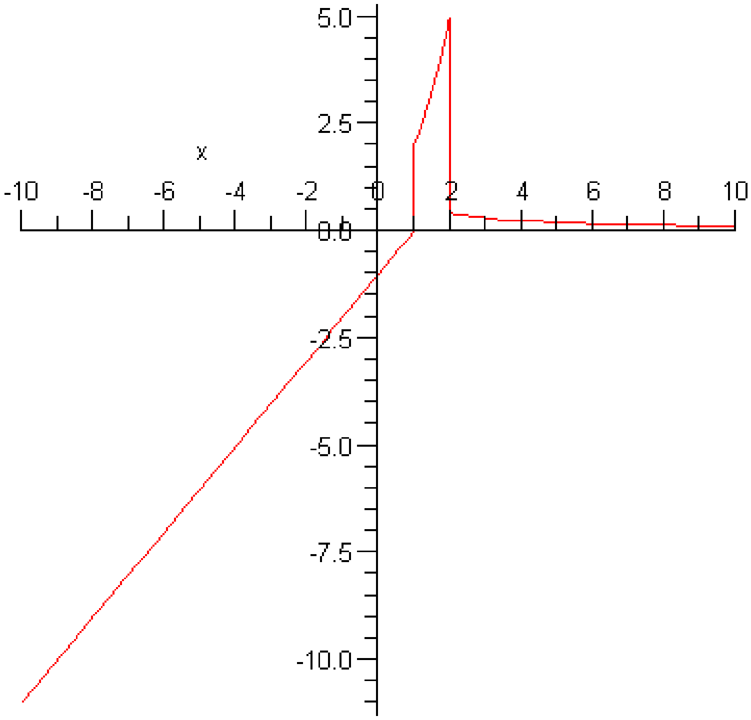
$$f(x) = x^2 + 1, -1 < x < 2$$

$$f(x) = \frac{x}{x^2 + 1}, 2 \leq x.$$

$$\left[\begin{array}{cc} x & f(x) \\ -3 & -4 \\ -2 & -3 \\ -1 & -2 \\ -\frac{1}{2} & \frac{5}{4} \\ 0 & 1 \\ 1 & 2 \\ 3 & \frac{3}{10} \\ 4 & \frac{4}{17} \\ 2 & \frac{2}{5} \end{array}\right]$$



Plot of points



Graph of f

Of course the vertical line segments do NOT belong to the graph, they appear only because the software being used by your beloved instructor put them there. But the old guy knows that they are NOT part of the graph. Also $(-1,-2),(2,\frac{2}{5})$ are in the graph but not $(-1,2),(2,5)$.